Complex Geometry and Topology

Jonas Stelzig

I work at the interface of complex geometry and topology. There are some famously hard but relatively simple to state questions in this area. For example:

1. Is there any almost complex compact manifold[1] of real dimension ≥ 6 which does not admit the structure of a complex manifold? Most notoriously, this is open for $X = S^6$, but our ignorance goes far beyond that case.

2. Is there any compact complex manifold of dimension ≥ 3 which admits only Kähler complex structures? E.g. it is not known if there is any 'exotic' complex structure on $\mathbb{CP}^3$ (but it would need to be non-Kähler).

Fortunately for any prospective PhD student (and everyone else in the field), there are also many other open and interesting questions which are more approachable! Any project with me would likely involve thinking about the various cohomology groups and/or (rational) homotopy theory of (mostly compact) complex manifolds and their underlying topological spaces. The precise plan would best be settled with the prospective student to adapt it to their background and interests, but some concrete directions may be the following:

1. **Additive cohomological invariants:** To any compact complex manifold X, one can associate a bunch of numbers, e.g. the dimension of various cohomology groups, $b_k(X) = \dim H^k_{dR}(X)$, $h^{p,q}_{\bar{\partial}}(X) := \dim H^{p,q}_{\bar{\partial}}(X)$, $h^{p,q}_{BC}(X) := \dim H^{p,q}_{BC}(X)$, ... or Chern numbers, e.g. $c_n(X) := \int_X c_n(TX)$. Some fundamental questions about these numbers, going back to Hirzebruch [Hir54] are: Which relations hold universally, i.e. on any complex manifold of a given dimension, and which linear combinations are preserved under natural equivalence relations on compact complex manifolds (homeomorphism, diffeomorphism, deformation equivalence,...)? Restricted to Hodge, Betti and Chern numbers, the first, and the second for homeomorphisms and diffeomorphisms, have been answered in a series of papers [Kot92], [Kot97], [Kot12], [KS13], [Ste22]. For 'all' numbers, and including deformation invariance, they are still open. Another, related direction is to consider not only linear relations, see e.g. [PS19]. A project in this direction would involve a combination of explicit constructions of complex manifolds with prescribed properties and algebraic calculations with very big rings that are used as book-keeping tools to track all these numbers.

2. **Pluripotential homotopy theory** To any manifold, one can associate the commutative differential graded algebra (short: cdga) of differential forms $(\mathcal{A}(X), d, \wedge)$. In rational homotopy theory [Sul77], [FHT01], one models manifolds (and more generally spaces) by the cdga of (polynomial) differential forms. In good situations[2], one can recover the "homotopy theory up torsion" from the quasi-isomorphism class of the cdga of forms. E.g. one can read of the ranks of the free parts of the higher homotopy groups. Moreover, the quasi-isomorphism class of any cdga can be represented uniquely by a so-called minimal model, which is computable by an explicit algorithm. There are long-standing interesting connections to geometry, e.g. it is a classical result [DGMS75] that compact

[1] I.e. a smooth compact manifold for which the tangent bundle TX admits the structure of a complex vector bundle refining the one as a real vector bundle. This is a 'topological' property.

[2] for finite type simply connected, or more generally nilpotent, spaces

Kähler manifolds are ‘formal’, meaning the cdga of forms is quasi-isomorphic to that of their cohomology. As a consequence, the (free parts of) the homotopy groups of simply connected compact Kähler manifolds can be computed from the cohomology ring alone.

Now, on any complex manifold, the algebra of forms has the extra structure of a bigrading for which the differential $d = \partial + \bar{\partial}$ splits into components of type $(1,0)$ and $(0,1)$. We call this structure commutative bigraded, bidifferential algebra (short: cbba). Recall that a quasi-isomorphism of cdga's can be understood as a map which tracks information on existence and essential uniqueness of solutions to the ‘potential’-equation

$$dx = y$$

For cbba's can then define a stronger notion of quasi-isomorphism that tracks information on existence and uniqueness to the ‘pluripotential’ equation

$$\partial\bar{\partial}x = y$$

This gives rise to a theory which on a formal level behaves very much in analogy to rational homotopy theory, but which encodes invariants of complex structures[Ste25]. In this new theory, compact Kähler manifolds are no longer formal [MS24][PSZ24] and one gets new invariants even for projective manifolds. There are many projects here that start from learning about a ‘well-known’ result in rational homotopy theory and work out to which extent or in what form it generalizes to the pluripotential setup and what that tells us about complex manifolds.

If you are interested in one of these, or any other project with me, having taken courses in some (but not necessarily all of) the following topics is useful: Complex (differential) geometry, algebraic geometry, algebraic topology.

Note: I am transitioning from LMU Munich to Stockholm University and I will be fully present (and formally employed) in Stockholm only from January 2027 onwards. Because of that, in the last months of 2026 the supervision from my part would be mostly remote (but there would be a second supervisor on site and I may be around in person a few times as well).

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