Effective Characterization of Quasi-Abelian Surfaces

Joint Work with M. Mendes Lopes and R. Pardini

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Plan of the talk

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1. Iitaka's Philosophy
2. Quasi-abelian (semi-abelian) varieties
3. Characterization of quasi-abelian varieties
4. Strategy

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Section 1
Iitaka's Philosophy

A tale of two worlds

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Projective World

X smooth **complex** projective variety of dimension n

A tale of two worlds

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Projective World

X smooth **complex** projective variety of dimension n

Quasi-Projective World

V smooth quasi-projective variety of dimension n

Hironaka smooth.

$V \hookrightarrow X$ proje

$X \setminus V = D$ snc.

A tale of two worlds

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Projective World

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Quasi-Projective World

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$\Updownarrow$

(X, D) with D a *snc* divisor

A tale of two worlds

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Projective World

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Ω_X

the sheaf of regular 1-forms

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$\Updownarrow$

(X, D) with D a *snc* divisor

$\Omega_X(\log D)$

the sheaf of logarithmic 1-forms

$p \in X$ $z_1 \dots z_n$ local system $\{0 = z_1 \cdots z_s = 0\} = D$

$$\Omega_X(\log D)_p = \sum_{i=1}^{s} \theta_{X,p} \frac{dz_i}{z_i} + \sum_{i=s+1}^{n} \theta_{X,p}\, dz_i$$

Meromorphic forms on X with logarithmic poles along D

A tale of two worlds

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Projective World

X smooth **complex** projective variety of dimension n

Vb. of rank n

Ω_X

the sheaf of regular 1-forms

$\omega_X := \bigwedge^n \Omega_X$ the canonical sheaf

Quasi-Projective World

V smooth quasi-projective variety of dimension n

$\Updownarrow$

(X, D) with D a *snc* divisor

$\Omega_X(\log D)$

the sheaf of logarithmic 1-forms

A tale of two worlds

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Projective World	Quasi-Projective World
X smooth **complex** projective variety of dimension n	V smooth quasi-projective variety of dimension n
	$\Updownarrow$
	(X, D) with D a *snc* divisor
Ω_X	$\Omega_X(\log D)$
the sheaf of regular 1-forms	the sheaf of logarithmic 1-forms
$\omega_X := \bigwedge^n \Omega_X$ the canonical sheaf	$\bigwedge^n \Omega_X(\log D) \simeq \mathcal{O}_X(K_X + D)$ the log-canonical sheaf

K_X canonical divisor of X

Logarithmic invariants

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Projective World	Quasi-Projective World
X smooth projective	V smooth quasi-projective $\Leftrightarrow$ (X, D)

Logarithmic invariants

Stockholm University

Projective World	Quasi-Projective World
X smooth projective	V smooth quasi-projective $\Leftrightarrow$ (X, D)
$q(X) := h^0(X, \Omega_X)$ the irregularity of X	
$P_m(X) := h^0(X, \omega_X^{\otimes m})$ the m-th plurigenus	
$\kappa(X)$ the Kodaira dimension	

Stable dimension

$\varphi_{\omega_X^{can}}(X)$

Logarithmic invariants

Projective World	Quasi-Projective World
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Logarithmic invariants

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$\kappa(X, \mathcal{O}_X(K_X + D))$

Logarithmic invariants

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$\kappa(X)$ the Kodaira dimension	$\overline{\kappa}(V)$ the log-Kodaira dimension

- The logarithmic invariants do not depend on the compactification X.
- They are not birational invariants.

$V = \mathbb{P}^1 \setminus \{3 \text{ pt}\}$ $\longleftrightarrow$ $\mathbb{P}^1$ birational equivalent $D = \emptyset$

$\overline{K}(V) = 1$

$\overline{P_m}(V) \geq 0 \quad m > 0$

$\overline{q}(V) = 2$

$\overline{K}(\mathbb{P}^1) = \kappa(\mathbb{P}^1) = -\infty$

$\overline{P_m}(\mathbb{P}^1) = 0$ if $m > 0$

$\overline{q}(\mathbb{P}^1) = 0$

Iitaka's Philosophy

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Iitaka's Philosophy

To any statement in the projective world that is dictated by the behavior of regular forms there should be a corresponding statement in the quasi-projective world dictated by the behavior of logarithmic forms.

Iitaka's Philosophy

Stockholm University

Iitaka's Philosophy

To any statement in the projective world that is dictated by the behavior of regular forms there should be a corresponding statement in the quasi-projective world dictated by the behavior of logarithmic forms.

Log-CastelnuovoCriterion (Zhu 2014)

The following statement are equivalent for a smooth quasi-projective surface V:

1. V is log-rationally connected;
2. $h^0(X, \Omega_X(\log D)^{\otimes m}) = 0$ for any $m \geq 1$;
3. $\overline{\kappa}(V) = -\infty$ and $h^0(X, S^{12}\Omega_X(\log D)) = 0$,

Log-rationally connected = every 2 points lie on a curve of log-genus 0, that is either a $\mathbb{P}^1$ or an $\mathbb{A}^1$

Today's topic

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We look for a quasi-projective analogue of the following statement:

Theorem (Enriques 1905, Chen–Hacon 2001)

Let S be a smooth complex projective surface such that

$$P_1(S) = P_2(S) = 1, \quad q(S) = 2.$$

Then S is birationally equivalent to an abelian surface.

Today's topic

Enrique $P_1(S) = P_4(S) = 1$

We look for a quasi-projective analogue of the following statement:

Theorem (Enriques 1905, Chen–Hacon 2001)

Let S be a smooth complex projective surface such that

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Then S is birationally equivalent to an abelian surface.

Show that $alb_S : S \longrightarrow Alb(S)$ is birational
↳ constructed by integrating 1-forms

Questions:

- Why is this a good candidate for Iitaka's philosophy?
- What is the right notion of equivalence to consider?
- What is the analogue of an abelian variety?

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Section 2
Quasi-abelian varieties

The tale continues

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Projective World

A abelian variety
$A \simeq \mathbb{C}^n/\Lambda$ with Λ a free abelian group of rank $2n$ + Riemann conditio

$$\Omega_A \simeq \mathcal{O}_A^{\oplus n}$$

$$\omega_A \simeq \mathcal{O}_A$$

$\kappa(A) = 0$
$P_m(A) = 1$ for every $m \geq 0$
$q(A) = n$

Quasi-Projective World

The tale continues

Stockholm University

Projective World

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Quasi-Projective World

G quasi-abelian variety
is an algebraic group sitting in SES

$$0 \to G_m^r \to G \to A \to 0$$

$\downarrow$
abelian variety.

The tale continues

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Projective World	Quasi-Projective World
A abelian variety $A \simeq \mathbb{C}^n/\Lambda$ with Λ a free abelian group of rank $2n$	G quasi-abelian variety $G \simeq \mathbb{C}^n/L$ with L a free abelian group of rank $\leq 2n$
$\Omega_A \simeq \mathcal{O}_A^{\oplus n}$	
$\omega_A \simeq \mathcal{O}_A$	
$\kappa(A) = 0$ $P_m(A) = 1$ for every $m \geq 0$ $q(A) = n$	

$$0 \to \mathbb{G}_m^r \to G \to A \to 0$$

$$\mathbb{P}^r \longrightarrow Z$$

$\exists$ a compactification $\bar{G}, Z$ that is a $\mathbb{P}^r$-bundle over A such that $\Delta = Z \cdot G$ is snc $\quad \Omega_Z^1(\log \Delta) \simeq \mathcal{O}_Z^{\oplus n}$

$n=2$

$0 \to \mathbb{G}_m^v \to G \to A \to 0$

$\dim A = 2$ $\quad G \simeq A \quad A = Z \quad \Delta = \emptyset$

$\dim A = 1$ $\quad Z = \mathbb{P}_A(\mathcal{O}_A \oplus L) \quad L \in \mathrm{Pic}^\circ(A)$

$\Delta = \Delta_1 \cup \Delta_2$ two sections.

$\dim A = 0 \quad G \simeq \mathbb{G}_m^2 \quad Z = \mathbb{P}^1 \times \mathbb{P}^1$

$\Omega_Z(\log \Delta) = \mathcal{O}_Z \oplus \mathcal{O}_Z$ $\quad \Delta =$ ⧦

$\omega_Z(\Delta) \simeq \mathcal{O}_Z$

The tale continues

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Projective World

A abelian variety
$A \simeq \mathbb{C}^n/\Lambda$ with Λ a free abelian group of rank $2n$

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Quasi-Projective World

G quasi-abelian variety
$G \simeq \mathbb{C}^n/L$ with Λ a free abelian group of rank $\leq 2n$

$$\Omega_Z(\log \Delta) \simeq \mathcal{O}_Z^{\oplus n}$$

The tale continues

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Projective World	Quasi-Projective World
A abelian variety $A \simeq \mathbb{C}^n/\Lambda$ with Λ a free abelian group of rank $2n$	G quasi-abelian variety $G \simeq \mathbb{C}^n/L$ with Λ a free abelian group of rank $\leq 2n$
$\Omega_A \simeq \mathcal{O}_A^{\oplus n}$	$\Omega_Z(\log \Delta) \simeq \mathcal{O}_Z^{\oplus n}$
$\omega_A \simeq \mathcal{O}_A$	$\mathcal{O}_Z(K_Z + \Delta) \simeq \mathcal{O}_Z$
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The tale continues

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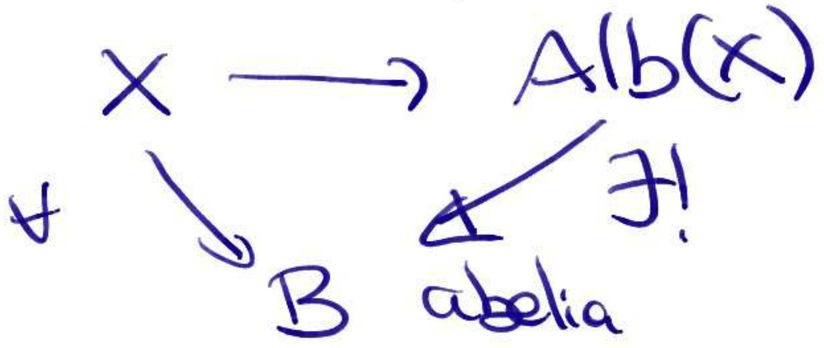
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$\kappa(A) = 0$ $P_m(A) = 1$ for every $m \geq 0$ $q(A) = n$	$\overline{\kappa(G)} = 0$ $\overline{P}_m(G) = 1$ for every $m \geq 0$ $\overline{q}(G) = n$

The quasi-Albanese variety

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Projective World

Given a smooth projective variety X, there is an abelian variety $\mathrm{Alb}(X)$, and a morphism $a_X : X \to \mathrm{Alb}(X)$ satisfying the obvious universal property. We call the pair $(\mathrm{Alb}(X), a_X)$ the Albanese variety of X.



Quasi-Projective World

The quasi-Albanese variety

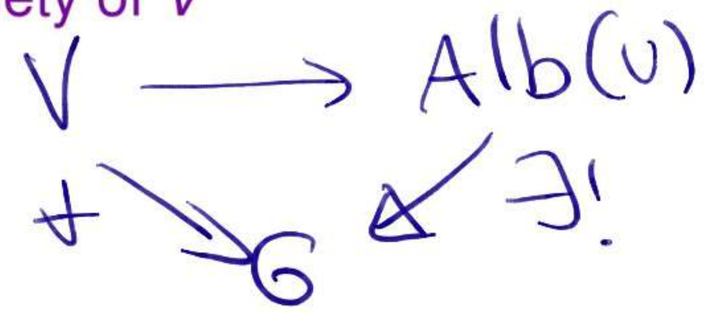
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Quasi-Projective World

Given a smooth quasi-projective variety V, there is a quasi-abelian variety $\mathrm{Alb}(V)$, and a morphism $a_V : V \to \mathrm{Alb}(V)$ satisfying the obvious universal property. We call the pair $(\mathrm{Alb}(V), a_V)$ the quasi-Albanese variety of V



The quasi-Albanese variety

Stockholm University

Projective World

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Remark

In both cases it is constructed by integrating 1-forms over 1-cycles. It still fit Iitaka's Philosophy

$q = q(X)$
$\bar{q} = \bar{q}(V)$
$r = \bar{q} - q$

$H^0(X, \Omega_X(\log D))$
$= \langle \underbrace{\omega_1 \cdots \omega_q}_{\text{basis for } H^0(X, \Omega_X)} \; \omega_{q+1} \cdots \omega_{\bar{q}} \rangle$

$i : V \hookrightarrow X$

$H_1(V, \mathbb{Z})_{tf} = \langle \underbrace{\alpha_1 \cdots \alpha_{2q}}_{i_*(\;)\text{ basis } H_1(X, \mathbb{Z})_{tf}} \; \underbrace{\beta_1 \cdots \beta_r}_{i_*(\beta_i) = 0} \rangle$

$$A_i := \left(\int_{\alpha_i} \omega_j \right)_{j=1 \dots \bar{q}} \in \mathbb{C}^{\bar{q}}$$

$$B_i := \left(\int_{\beta_i} \omega_j \right)_{j=1 \dots \bar{q}}$$

linearly indep.

$$L := \sum_{\mathbb{Z}} \mathbb{Z} A_i + \sum_{\mathbb{Z}} \mathbb{Z} B_i \qquad Alb(V) := \mathbb{C}^{\bar{q}} / L$$

Fix $p_0 \in V$

$$\begin{array}{ccc} V & \longrightarrow & Alb(V) \\ p & \longmapsto & \left(\int_{p_0}^{p} \omega_1 \dots \int_{p_0}^{p} \omega_{\bar{q}} \right) \end{array}$$

$$0 \to G_M^r \to Alb(V) \to \underline{Alb(X)} \to 0$$

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Section 3
Characterization of quasi-abelian varieties

Iitaka's characterization

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Theorem (Iitaka 1979)

Let V be a smooth complex quasi-projective surface satisfying

$$\overline{\kappa}(V) = 0, \quad \overline{q}(V) = 2.$$

Then $a_V : V \to \mathrm{Alb}(V)$ is birational. Furthermore there are finitely many points $p_1, \dots p_k$ in $\mathrm{Alb}(V)$, and an open set $V_0 \subseteq V$ such that $a_{V|_{V_0}} : V_0 \to \mathrm{Alb}(V) \backslash \{p_1, \dots, p_k\}$ is proper.

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Stockholm University

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- In the language of Iitaka, a_V is a WWPB equivalence.

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  WWPB= "Weak Weak Proper Birational"

Iitaka's characterization

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- In the language of Iitaka, a_V is a WWPB equivalence.
  WWPB= "Weak Weak Proper Birational"
- WWPB equivalences preserve the logarithmic invariants.
- WWPB equivalences between affine varieties are isomorphisms.

Main Result

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Theorem (Mendes Lopes, Pardini, :-))

Let V be a smooth complex quasi-projective surface with $\overline{q}(V) = 2$. Assume that either one of the following hold:

1. $\overline{P}_1(V) = \overline{P}_2(V) = 1$, and $q(X) > 0$;
2. $\overline{P}_1(V) = \overline{P}_3(V) = 1$, and $q(X) = 0$.

Then $a_V : V \to \mathrm{Alb}(V)$ is a WWPB equivalence.

Corollary

If V is an affine surface with $\overline{P}_1(V) = \overline{P}_3(V) = 1$ and $\overline{q}(V) = 2$, then V is isomorphic to $\mathbb{G}_m^2$

Example

Let X' be a $\mathbb{Z}/3$-cyclic cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on 3 fibers of each fibration.

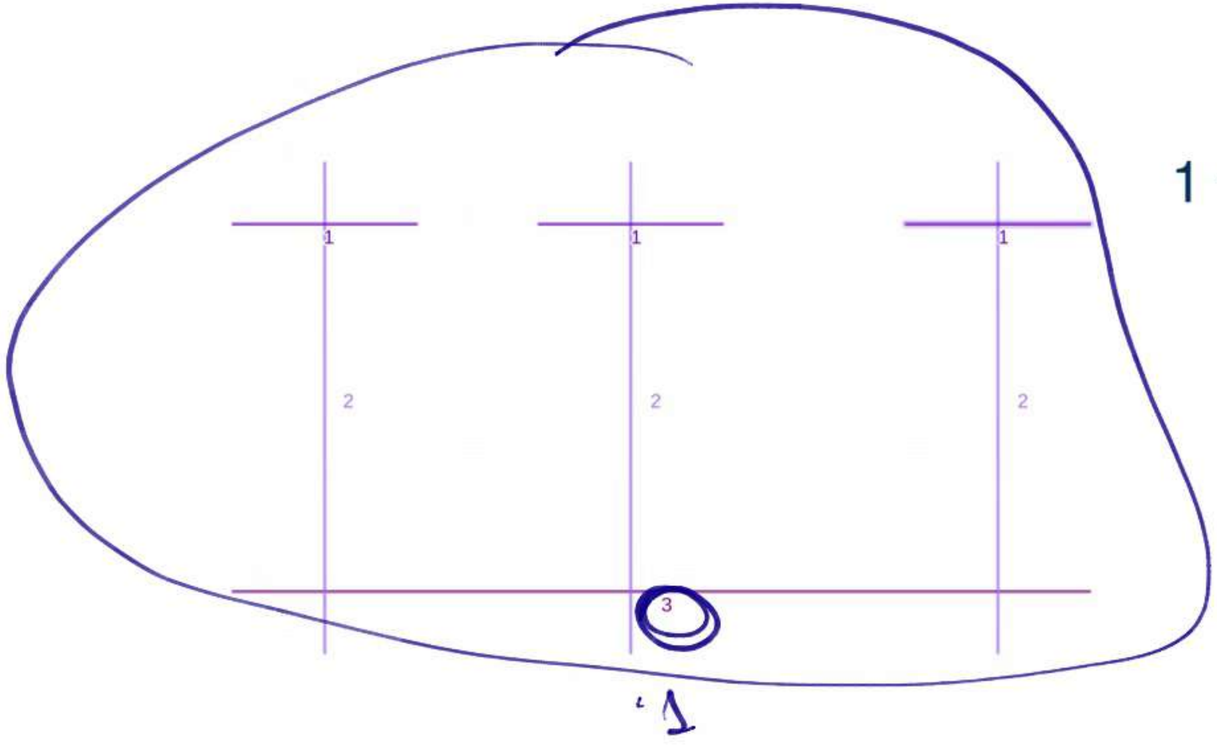
$X \longrightarrow X'$ resolution of singularities

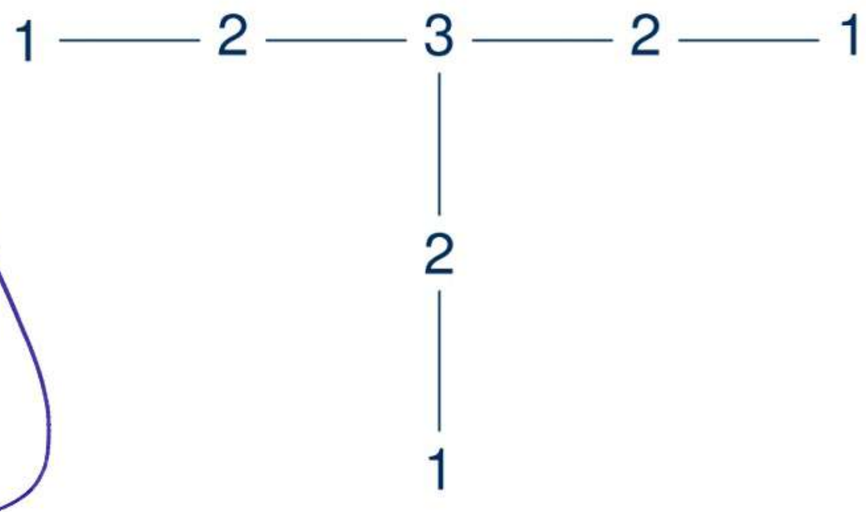
Then X is an elliptic $K3$ surface such that the fibration $X \to \mathbb{P}^1$ has 3 fibers of type IV*, F_1, F_2, and F_3.

Example

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Then X is an elliptic $K3$ surface such that the fibration $X \to \mathbb{P}^1$ has 3 fibers of type IV*, F_1, F_2, and F_3.





Example

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Denote by F_i^s the support of the fiber F_i. Let

$$V := X \backslash \{F_1^s,\ F_2^s,\ F_3^s\}.$$

We have that

- $\overline{P}_1(V) = \overline{P}_2(V) = 1$,
- $\overline{q}(V) = 2$,
- $q(X) = 0$, and...

Example

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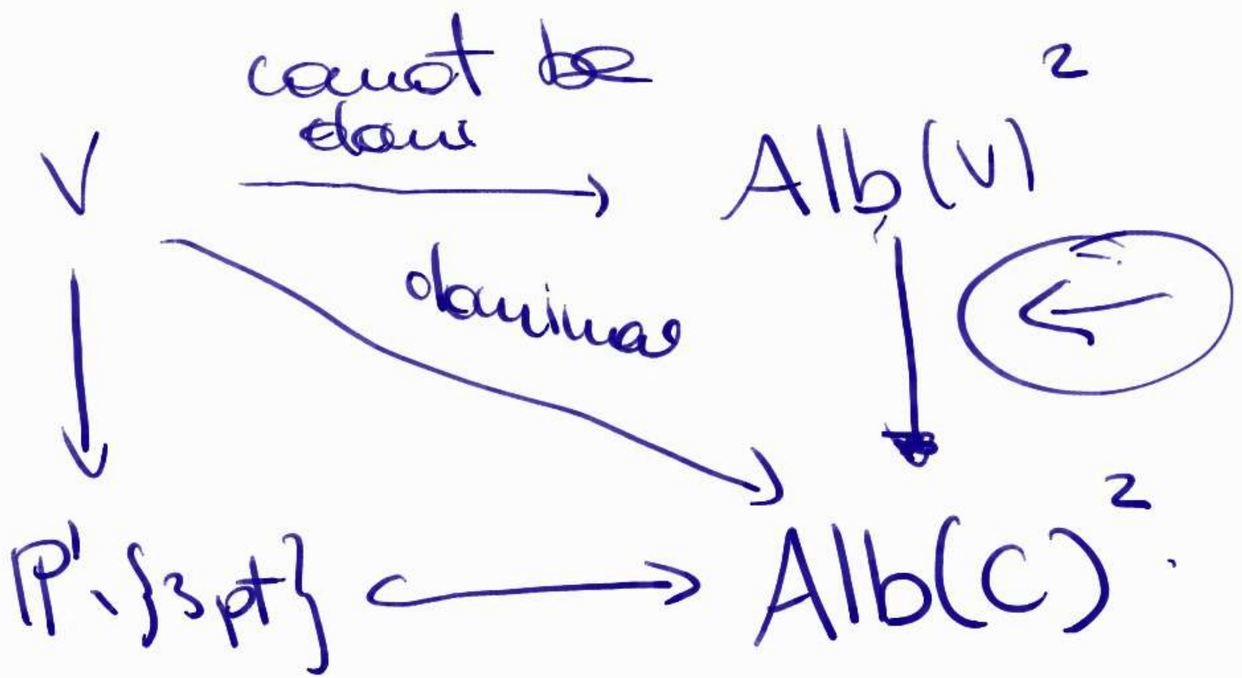
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- $\overline{P}_1(V) = \overline{P}_2(V) = 1$,
- $\overline{q}(V) = 2$,
- $q(X) = 0$, and...

...the quasi-Albanese morphism is not dominant!!!



$$\langle \mathbb{P}^1 \setminus \{3 pt\} \rangle = Alb(C)$$

Higher dimension?

Higher dimension?

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Theorem (Kawamata 1981)

Let V be a quasi-projective variety with

$$\overline{\kappa}(V) = 0, \quad \overline{q}(V) = \dim V.$$

Then the quasi-Albanese morphism $a_V : V \to \mathrm{Alb}(V)$ is birational.

correctly working. WWPB

Higher dimension?

Stockholm University

Theorem (Kawamata 1981)

Let V be a quasi-projective variety with

$$\overline{\kappa}(V) = 0, \quad \overline{q}(V) = \dim V.$$

Then the quasi-Albanese morphism $a_V : V \to \mathrm{Alb}(V)$ is birational.

Effective versions of this result using the first 2 plurigenera were given by Chen–Hacon in 2001 (when V is projective) and Pareschi–Popa–Schnell in 2014 (when V is compact Kähler).

Conjectures

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(Very^n Bold) Conjecture

Let V be a quasi-projective variety with

$$\overline{P}_1(V) = \overline{P}_3(V) = 1, \quad \overline{q}(V) = \dim V.$$

Then the quasi-Albanese morphism $a_V : V \to \mathrm{Alb}(V)$ is a WWPB equivalence.

With n as big as you want

Conjectures

Stockholm University

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Let V be a quasi-projective variety with

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Then the quasi-Albanese morphism $a_V : V \to \mathrm{Alb}(V)$ is a WWPB equivalence.

With n as big as you want

Okay Conjecture

Let V be a quasi-projective variety. Then there exist k a positive integer independent of the dimension of V such that, if

$$\overline{P}_1(V) = \overline{P}_k(V) = 1, \quad \overline{q}(V) = \dim V,$$

the quasi-Albanese morphism $a_V : V \to \mathrm{Alb}(V)$ is a WWPB equivalence.

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Section 4
Strategy

Outline of the proof

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There are three main steps:

- the quasi-Albanese morphism is dominant;
- the quasi-Albanese morphism is birational;
- the quasi-Albanese morphism is a WWBP equivalence.

Case $q(X) = 2$

- The first two points follows (almost) directly from Chen–Hacon, with the help of generic vanishing (and recent extensions to pairs).
- It remains to show that the Albanese morphism of X contracts the boundary D.

$\exists$ sheaf on a projective variety X

i-th cohomological jump locus

$$V^i(X,\mathcal{F}) := \{\alpha \in \mathrm{Pic}^0(X) \mid h^i(X,\mathcal{F}\otimes\alpha) \neq 0\}$$

$\mathcal{F}$ is GV $\quad$ codim $V^i(X,\mathcal{F}) \geq i \quad \forall i>0$

PROPAGATION $\mathcal{F}$ is GV $\quad W \subseteq V^0(X,\mathcal{F})$

cp of codimension $k \Rightarrow$ $\boxed{W \subseteq V^k(X,\mathcal{F})}$

is a component

Pepe - Schnell $\quad \omega_X(D)$ is GV $\quad D$ snc.

Shibata $\quad V^0(X, \omega_X(D)) =$ union of torsion translate by subtori

① Same argument by Ein-Lazarsfeld

$\overline{P_1(V)} = \overline{P_2(V)} = 1 \Rightarrow \{\mathcal{O}_X\} \subseteq V^0(X, \omega_X(D))$

isolated

$V^0(Alb(X), \underbrace{a_{X*}(\omega_X D)}_{GV.})$

$\mathcal{O}_X \subseteq V^q(X, a_{X*}\omega_X D)$

$\Rightarrow a_X$ is surjective



$q(X) = 2$

$P_1(X) > 0$

$P_1(V) \geq P_1(X) = 1 > 0$

$P_2(V) \geq P_2(X) = 1 > 0$

a_X is birational

a_V is generically deg 1 CH

Case $q(X) = 1, 0$

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- The argument for the dominance of the quasi-Albanese morphism proceeds smoothly thanks to generic vanishing when $q(X) = 1$.

Case $q(X) = 1, 0$

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Case $q(X) = 1, 0$

- The argument for the dominance of the quasi-Albanese morphism proceeds smoothly thanks to generic vanishing when $q(X) = 1$.
- You do not really want to know the argument in the case $q(X) = 0$...
  A great tool is the following formula

$$h^0(X, K_X + D) = p_a(D) + p_g(X) - q(X) + h^1(X, -D).$$

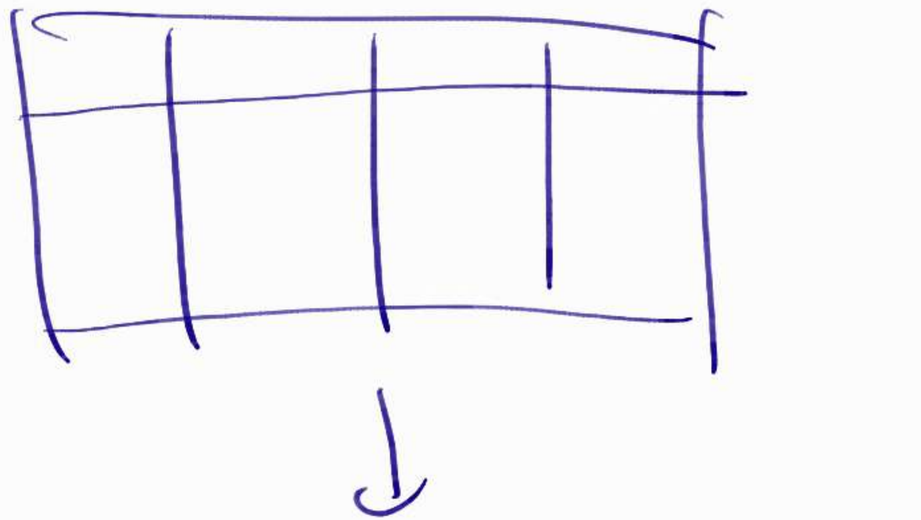
- Once one knows that the quasi-Albanese morphism is dominant (and hence generically finite), we can use the logarithmic ramification formula

$$K_X + D \sim \overline{R}_g$$

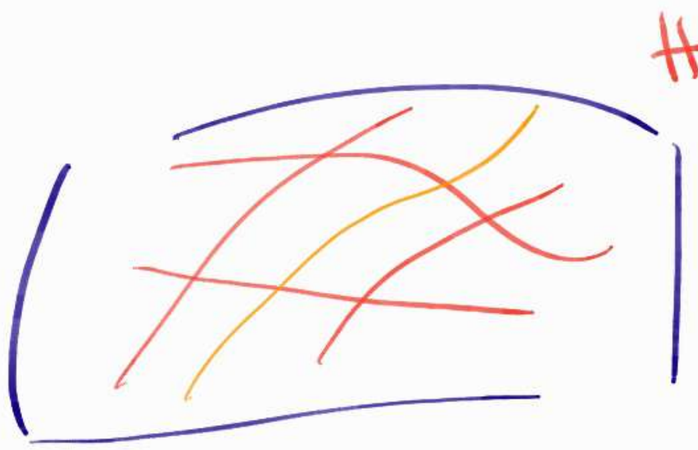
Assume that ω is not dominal

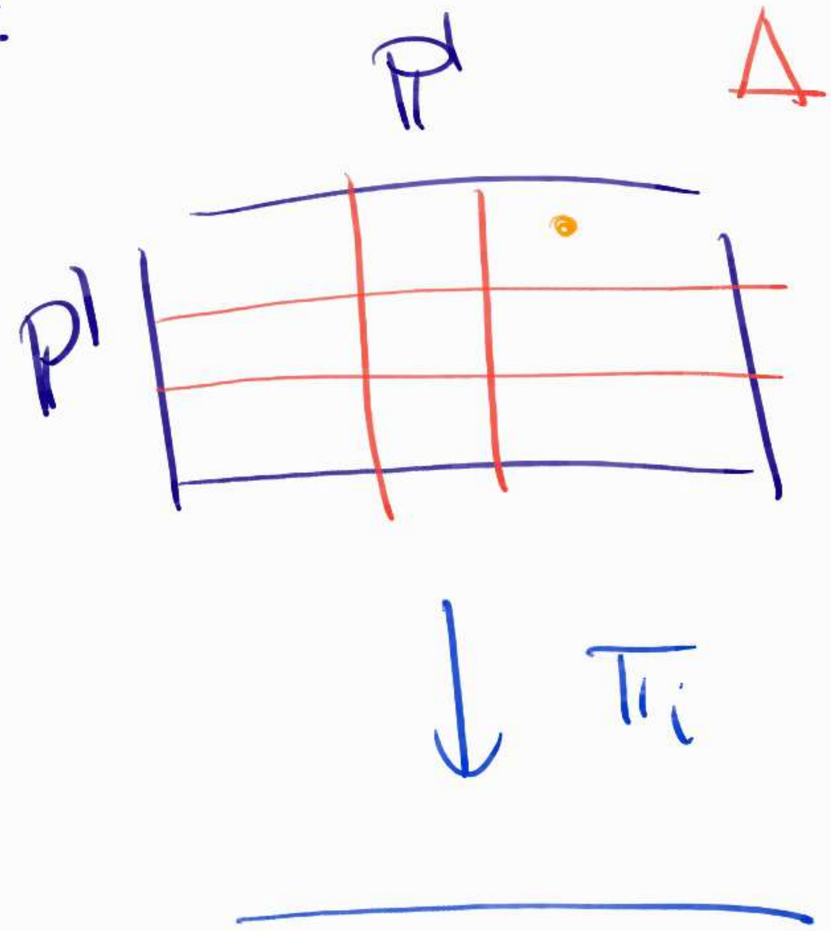
$$\begin{array}{ccc} V & \hookrightarrow & X \\ \downarrow & & \downarrow \\ C & \hookrightarrow & \mathbb{P}^1 \end{array}$$

$C = \mathbb{P}^1 \setminus \{3 \text{ pt}\}$



q_w Dominant $\Rightarrow$ gf.





$K_x + D = \bar{a}_v^* (K_z + \Delta) + \overline{R}\bar{a}_v$

all the α of $\overline{R}_g$ that are not 0

I take in t are contracted

$\rightsquigarrow K(R_{\bar{a}_v}) = 0$

Thank You!!!



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Section 5
WWPB

WPB

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Definition

A birational map $\varphi : V \dashrightarrow W$ is a WPB-equivalence if it can be written as a composition of

- proper birational morphism and their inverse;
- open immersions $V \subset V'$ such that $V' \setminus V$ has codimension at least 2, and their inverses.

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WPB-equivalent varieties have the same logarithmic plurigenera and irregularity. Therefore, it would seem that WPB-equivalence is the right equivalence to consider when studying the birational geometry of open varieties.

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Problem

the set of WPB-maps is not saturated.

WWPB

$$\mathcal{W} := \{f : V \dashrightarrow W \mid \text{there is } U \text{ and } g \text{ or } h \text{ such that either } f \circ g \text{ or } h \circ f \text{ are WPB}\},$$

Definition

A rational map $f : V \dashrightarrow W$ between algebraic varieties is a WWPB equivalence, if we can write $f = f_1 \circ \cdots \circ f_k$ with f_i birational maps such that either f_i or f_i^{-1} is in $\mathcal{W}$.

It can be proven that WWPB-equivalent varieties have the same plurigenera and the same irregularity.

Thank You!!!

