SPECIMEN ANALYTICUM,

THEOREMATA QUÆDAM NOVA

DE INTEGRALIBUS DEFINITIS, SUMMATIONE SERIERUM
EARUMQUE IN ALIAS SERIES TRANSFORMATIONE
EXHIBENS,

CUJUS PARTEM PRIMAM

VENIA AMPL. FAC. PHILOS. UPSAL.

P. P.

MAG. CAROLUS JOHANNES MALMSTEN
MATH. INFER. DOCENS

ET

THURE AUGUSTUS ALMGREN
Vestrog.

IN AUDITORIO GUSTAV. DIE VI APR. MDCCCXLII

H. A. M. S.

UPSALIAE,
EXCUDEBANT REGIAE ACADEMIAE TYPOGRAPHI.

Den Huldaste Moder

helgadt

af

sonlig vördnad, kärlek och tacksamhet.

Ex demonstratione, quæ duobus in locis a Cel. Liouville proposita est[1]), inter omnes constat, datam φ functionem ipsius x in exponentialem seriem hujusce formæ

$$\varphi(x) = S.A_m e^{mx} \quad \ldots \quad (1)$$

evolvi posse, ubi $S.A_m e^{mx}$ summam designat

$$A_{m_1} e^{m_1 x} + A_{m_2} e^{m_2 x} + A_{m_3} e^{m_3 x} + A_{m_4} e^{m_4 x} + \text{etc.}$$

Hæc formula (1) quantum jamjam in Analysin conduxerit, calculus ille satis indicio est, qui, nomine *Calcul des différentielles à indices quelquonques* celebratus, ab eodem Liouville inventus est[2]). Nunc nos in animum induximus, ut alium hujus formulæ usum ostenderemus, nimirum quomodo, inde proficiscentes, ad theoremata quædam nova et generalia pervenire possimus, quæ non minus in integralibus definitis cognoscendis, quam in summatione serierum earumque in alias series transformatione maximo sint usui.

1) Vide *Journ. de l'Ecole Polyt.* XXI. Cah. p. 4 et p. 74.

2) Vide *Journ. de l'École Polyt.* XXI. Cah.

§. 1. Ut vero ea, quæ proposita erunt, quam maxime generalia sint, primum nobis est demonstrandum, quomodo formulas, cum illa (1) congruas, habeamus, respectu ad functiones duarum pluriumve variabilium habito. Sit igitur $\varphi(x,y)$ functio quædam duarum variabilium independentium x et y, quam tamen ut functionem solius x primum consideremus. Tunc ex formula (1) sequitur

$$\varphi(x,y) = S.Y_m e^{mx} \quad \ldots\ldots\ldots \quad (5)$$

ubi Y_m, seu terminus generalis coëfficientium, functio est elementi y. Fit autem ex eadem formula (1)

$$Y_m = S.A_{m,n} e^{ny},$$

ubi $A_{m,n}$ coëfficientem ipsius e^{ny} in evolutione functionis Y_m designat; quod quidem, in (2) ductum, suppeditat

$$\varphi(x,y = S.S.A_{m,n} e^{mx+ny},$$

seu, brevitatis caussa,

$$\varphi(x,y) = S.A_{m,n} e^{mx+ny} \quad \ldots\ldots\ldots \quad (3)$$

Quod ad functiones plurium variabilium adtinet, facile perspicitur, eodem modo obtineri

$$\varphi(x,y,z..) = S.A_{m,n,p..} e^{mx+ny+pz+\text{etc.}}.$$

§. 2. Jam si in formula (3) $x+ru\sqrt{-1}$ et $y+su\sqrt{-1}$ pro ipsis x et y substituimus, fit

$$\varphi(x+ru\sqrt{-1}, y+su\sqrt{-1}) = S.A_{m,n} e^{mx+ny} e^{(mr+ns)u\sqrt{-1}}$$

et, mutatis ipsorum r et s signis,

$$\varphi(x-ru\sqrt{-1}, y-su\sqrt{-1}) = S.A_{m,n} e^{mx+ny} e^{-(mr+ns)u\sqrt{-1}}.$$

Hinc vero, si brevitatis caussa appellamus

$$\left.\begin{aligned} \psi(u) &= \frac{\varphi(x+ru\sqrt{-1}, y+su\sqrt{-1}) + \varphi(x-ru\sqrt{-1}, y-su\sqrt{-1})}{2} \\ f(u) &= \frac{\varphi(x+ru\sqrt{-1}, y+su\sqrt{-1}) - \varphi(x-ru\sqrt{-1}, y-su\sqrt{-1})}{2\sqrt{-1}} \end{aligned}\right\} \ldots (4)$$

addendo et subtrahendo proveniunt

$$\left.\begin{aligned} \psi(u) &= S.A_{m,n} e^{mx+ny} \operatorname{Cos}(mr+ns)u \\ f(u) &= S.A_{m,n} e^{mx+ny} \operatorname{Sin}(mr+ns)u. \end{aligned}\right\} \ldots\ldots (5)$$

§. 3. Primum jam in priori formularum (5) versemur, et utrimque multiplicemus illam per

$$\frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} du;$$

integratione inter $u = o$ et $u = \infty$ facta, obtinetur

$$\int_0^\infty \frac{\psi(u)(e^{au} - e^{-au})}{e^{\pi u} - e^{-\pi u}} du = S.A_{m,n} e^{mx+ny} \int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} \operatorname{Cos}(mr+ns)u \, . \, du$$

$$= S.A_{m,n} e^{mx+ny} \frac{e^{mr+ns} \operatorname{Sin} a}{1 + 2e^{mr+ns} \operatorname{Cos} a + e^{2(mr+ns)}}.$$

Cum vero sit

$$\frac{e^{mr+ns} \operatorname{Sin} a}{1 + 2e^{mr+ns} \operatorname{Cos} a + e^{2(mr+ns)}} = \sum_{i=1}^{i=\infty} (-1)^{i-1} e^{i(mr+ns)} \operatorname{Sin} ia,$$

et Σ et S nullo modo a se invicem pendeant, fit

$$\int_0^\infty \frac{\psi(u)(e^{au}-e^{-au})}{e^{\pi u}-e^{-\pi u}}du = \sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin} ia\, S.A_{m,n}e^{m(x+ir)+n(y+is)}.$$

Ex vero formula (3), si $x+ir$ et $y+is$ loco ipsorum x et y substituimus, eruitur

$$S.A_{m,n}e^{m(x+ir)+n(y+is)} = \varphi(x+ir, y+is),$$

quod denique, in formulam antecedentem ductum, suppeditat

$$\int_0^\infty \frac{\psi(u)(e^{au}-e^{-au})}{e^{\pi u}-e^{-\pi u}}du = \sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin} ia\,\varphi(x+ir, y+is) \ldots\ldots (6).$$

Hæc formula cum ex eo omnino pendeat, quod

$$S.A_{m,n}e^{mx+ny}\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\operatorname{Cos}(mr+ns)u\,.\,du$$

$$= S.A_{m,n}.\frac{e^{mr+ns}\operatorname{Sin} a}{1+2e^{mr+ns}\operatorname{Cos} a+e^{2(mr+ns)}},$$

facile apparet, nihil ex illa esse concludendum, quod contra id repugnet. Imprimis igitur, posito $a = p+q\sqrt{-1}$, hæc sunt observanda:

1:o Si $\varphi(x,y)$ ejusmodi sit functio, ut in

$$\varphi(x,y) = S.A_{m,n}e^{mx+ny}$$

m et n nusquam se imaginaria offerant, necesse est (id, quod sufficit) ut p, seu pars realis ipsius a, sit $< \pi$.

2:o Si vero alicubi in

$$\varphi(x,y) = S.A_{m,n}e^{mx+ny}$$

ipsis m et n formæ sint $\mu+\mu'\sqrt{-1}$ et $\nu+\nu'\sqrt{-1}$, atque maxima omnium μ', ν' sint $\mu_,'$, $\nu_,'$, necesse est ut

$$\mu_,'r+\nu_,'s+p<\pi;$$

quia tunc semper integrale hujus formæ inter cetera occurrit

$$\int_0^\infty \frac{e^{(\mu_,'r+\nu_,'s+p)u}\;e^{-(\mu_,'r+\nu_,'s+p)u}}{e^{\pi u}-e^{-\pi u}}\operatorname{Cos} vu\,.\,du,$$

quod in infinitum abit, nisi $\mu_,'r+\nu_,'s+p<\pi$.

§. 4. Ut applicationem quandam formulæ (6) ostendamus, sit

$$\varphi(x,y) = \operatorname{Log} x, \quad \text{unde} \quad \psi(u) = \tfrac{1}{2}\operatorname{Log}(x^2+r^2u^2);$$

posito $r=1$, quod nullo modo generalitatem minuit, obtinebimus

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\operatorname{Log}(x^2+u^2)du = 2\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}(x+i) \ldots (7)$$

Sit jam

$$L(a,x) = \int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\operatorname{Log}(x^2+u^2)du.$$

Ex relatione cognita

$$\left(\operatorname{Tang}\tfrac{1}{2}a - \frac{2y^x\operatorname{Sin} a}{1+2y\operatorname{Cos} a+y^2}\right) = 2\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin} ia\,(1-y^{x+i-1}),$$

si, multiplicatione per $\frac{dy}{\text{Log}\frac{1}{y}}$ utrimque facta, inter $y=0$ et $y=1$ integremus, habebimus

$$L(a,x)=\int_0^1 \left(\text{Tang}\tfrac{1}{2}a-\frac{2y^x\text{Sin}\,a}{1+2y\,\text{Cos}\,a+y^2}\right)\cdot\frac{dy}{\text{Log}\frac{1}{y}}\quad\ldots\quad(8)$$

quia in genere est

$$\int_0^1\frac{(1-y^n)dy}{\text{Log}\frac{1}{y}}=\text{Log}\,(n+1).$$

Facile vero demonstrari potest, si a in ratione qualibet commensurabili sit ad π, id est $a=\frac{m\pi}{n}$ (m et n numeri integri), integrale in dextero equationis (8) membro finite semper per functionem Γ exprimi posse. Nam sit, brevitatis caussa,

$$T=\int_0^1\left(\text{Tang}\tfrac{1}{2}a-\frac{2y^x\text{Sin}\,a}{1+2y\,\text{Cos}\,a+y^2}\right)\cdot\frac{dy}{\text{Log}\frac{1}{y}};$$

differentiatione respectu ipsius x facta, prodit

$$\frac{dT}{dx}=2\int_0^1\frac{y^x\text{Sin}\,a\,dy}{1+2y\,\text{Cos}\,a+y^2}.\quad\ldots\quad(9)$$

In suppositione vero $a=\frac{m\pi}{n}$ cognitum est integrale[3])

3) Vide *Legendre Exercises de Calc. Intégr.* Tom. II pag. 163 — 165. Observamus errorem, qui l. c. apud *Legendre* occurrit, ubi de formulis, a nobis in textu adhibitis, dicit, illam valere, cum m numerus impar, hanc vero, cum m numerus par est. At vero est statuendum, illam aut hanc valere, prout $m+n$ impar aut par sit.

$$\int_0^1\frac{y^x\text{Sin}\,a\,dy}{1+2y\,\text{Cos}\,a+y^2}=\frac{1}{dx}\cdot\sum_{i=1}^{i=n-1}(-1)^{i-1}\text{Sin}\,ia\left\{d\,\text{Log}\,\Gamma\left(\frac{x+n+i}{2n}\right)-d\,\text{Log}\,\Gamma\left(\frac{x+i}{2n}\right)\right\},$$

ubi $m+n$ numerus impar est, et

$$\int_0^1\frac{y^x\text{Sin}\,a\,dy}{1+2y\,\text{Cos}\,a+y^2}=\frac{1}{dx}\cdot\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\text{Sin}\,ia\left\{d\,\text{Log}\,\Gamma\left(\frac{x+n-i}{n}\right)-d\,\text{Log}\,\Gamma\left(\frac{x+i}{n}\right)\right\},$$

ubi $m+n$ numerus par est; quod quidem, in formulam (9) ductum, suppeditat

$$dT=2\sum_{i=1}^{i=n-1}(-1)^{i-1}\text{Sin}\,ia\left\{d\,\text{Log}\,\Gamma\left(\frac{x+n+i}{2n}\right)-d\,\text{Log}\,\Gamma\left(\frac{x+i}{2n}\right)\right\},$$

($m+n$ = num. imp.)

$$dT=2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\text{Sin}\,ia\left\{d\,\text{Log}\,\Gamma\left(\frac{x+n-i}{n}\right)-d\,\text{Log}\,\Gamma\left(\frac{x+i}{n}\right)\right\},$$

($m+n$ = num. par)

unde, integratione instituta,

$$\left.\begin{aligned}T&=C+2\sum_{i=1}^{i=n-1}(-1)^{i-1}\text{Sin}\,ia\,\text{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}\\ &\qquad(m+n=\text{num. imp.})\\ T&=C'+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\text{Sin}\,ia\,\text{Log}\left\{\frac{\Gamma\left(\frac{x+n-i}{n}\right)}{\Gamma\left(\frac{x+i}{n}\right)}\right\}\\ &\qquad(m+n=\text{num. par})\end{aligned}\right\}\quad\ldots\quad(10)$$

Hinc, si primum $x=r$ et deinde $x=s$ ponamus, subtrahendo obtinebimus hoc integrale, quod non antea fuit propositum,

$$\left.\begin{aligned}
&\int_0^1 \frac{y^r(1-y^{s-r})}{1+2y\,\mathrm{Cos}\,a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} \\
&\qquad = \mathrm{Cosec}\,a \sum_{i=1}^{i=n-1} (-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{s+n+i}{2n}\right)\Gamma\left(\frac{r+i}{2n}\right)}{\Gamma\left(\frac{r+n+i}{2n}\right)\Gamma\left(\frac{s+i}{2n}\right)}\right\} \\
&\qquad\qquad (m+n=\text{num. imp.}) \\
&\int_0^1 \frac{y^r(1-y^{s-r})}{1+2y\,\mathrm{Cos}\,a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} \\
&\qquad = \mathrm{Cosec}\,a \sum_{i=1}^{i=\frac{n-1}{2}} (-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{s+n-i}{n}\right)\Gamma\left(\frac{r+i}{n}\right)}{\Gamma\left(\frac{r+n-i}{n}\right)\Gamma\left(\frac{s+i}{n}\right)}\right\} \\
&\qquad\qquad (m+n=\text{num. par})
\end{aligned}\right\}. \quad (11)$$

Harum vero formularum ope valores etiam ipsorum C et C' jam determinari possunt. Nam, si primum $r=0$, $s=1$ et deinde $r=1$, $s=2$ ponamus, subtractione facta, invenimus

$$\int_0^1 \frac{\mathrm{Sin}\,a(1-2y+y^2)}{1+2y\,\mathrm{Cos}\,a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} = \sum_{i=1}^{i=n-1} (-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\left(\Gamma\left(\frac{n+i+1}{2n}\right)\right)^2\Gamma\left(\frac{i+2}{2n}\right)\Gamma\left(\frac{i}{2n}\right)}{\left(\Gamma\left(\frac{i+1}{2n}\right)\right)^2\Gamma\left(\frac{n+i}{2n}\right)\Gamma\left(\frac{n+i+2}{2n}\right)}\right\}$$

$(m+n=\text{num. imp.})$

$$\int_0^1 \frac{\mathrm{Sin}\,a(1-2y+y^2)}{1+2y\,\mathrm{Cos}\,a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} = \sum_{i=1}^{i=\frac{n-i}{2}} (-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\left(\Gamma\left(\frac{n+1-i}{n}\right)\right)^2\Gamma\left(\frac{i+2}{n}\right)\Gamma\left(\frac{i}{n}\right)}{\left(\Gamma\left(\frac{i+1}{n}\right)\right)^2\Gamma\left(\frac{n-i}{n}\right)\Gamma\left(\frac{n+2-i}{n}\right)}\right\},$$

$(m+n=\text{num. par})$

atque etiam ex formula (10), posito $x=1$, prodit

$$\int_0^1 \frac{\mathrm{Sin}\,a(1-2y+y^2)}{1+2y\,\mathrm{Cos}\,a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} = (1+\mathrm{Cos}\,a)\left\{C+2\sum_{i=1}^{i=n-1}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{n+i+1}{2n}\right)}{\Gamma\left(\frac{i+1}{2n}\right)}\right\}\right\}$$

$(m+n=\text{num. imp.})$

$$\int_0^1 \frac{\mathrm{Sin}\,a(1-2y+y^2)}{1+2y\,\mathrm{Cos}\,a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} = (1+\mathrm{Cos}\,a)\left\{C'+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{n+1-i}{n}\right)}{\Gamma\left(\frac{i+1}{n}\right)}\right\}\right\}$$

$(m+n=\text{num. par})$

quæ formulæ, inter se comparatæ, cum sit omnino

$$2\,\mathrm{Sin}\,ma\,\mathrm{Cos}\,a=\mathrm{Sin}(m+1)a+\mathrm{Sin}(m-1)a,$$

hos valores ipsorum C et C' suppeditant

$$C=\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,2n,$$

$$C'=\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,n.$$

Fit igitur, substitutione in (10) facta,

$$T=\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,2n+2\sum_{i=1}^{i=n-1}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}$$

$(m+n=\text{num. imp.})$

$$T=\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,n+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n-i}{n}\right)}{\Gamma\left(\frac{x+i}{n}\right)}\right\}$$

$(m+n=\text{num. par})$

atque idcirco etiam, ex formula (8), posito $a = \frac{m\pi}{n}$,

$$\left.\begin{aligned} L(a,x) &= \int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} \operatorname{Log}(x^2+u^2)\,du \\ &= \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} 2n + 2\sum_{i=1}^{i=n-1} (-1)^{i-1} \operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\} \\ &\qquad (m+n = \text{num. imp.}) \\ L(a,x) &= \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} n + 2\sum_{i=1}^{i=\frac{n-1}{2}} (-1)^{i-1} \operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i}{n}\right)}{\Gamma\left(\frac{x+i}{n}\right)}\right\} \\ &\qquad (m+n = \text{num. par}) \end{aligned}\right\} \quad (12)$$

Ponendo $x = 1$ et $a = \frac{\pi}{2}$, unde $m = 1$, $n = 2$, e priori accipies

$$\int_0^\infty \frac{\operatorname{Log}(1+u^2)\,du}{e^{\frac{1}{2}\pi u} - e^{-\frac{1}{2}\pi u}} = \operatorname{Log}\left(\frac{4}{\pi}\right), \quad . \quad . \quad . \quad . \quad (13)$$

unde, eadem suppositione in (7) facta, et comparatione rite instituta, cognita illa Wallisii formula prodit

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot \cdots}$$

§. 5. Sit jam in (12) $x = o$, et ponamus

$$e^{\frac{\pi u}{n}} = y, \text{ unde } u = \frac{n}{\pi} \operatorname{Log} y;$$

cum sit $a = \frac{m\pi}{n}$, obtinebis

$$\int_1^\infty \frac{y^{m-1} - y^{-m-1}}{y^n - y^{-n}} \left\{\operatorname{Log}\frac{n}{\pi} + \operatorname{Log}(\operatorname{Log} y)\right\} dy$$

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$$= \frac{\pi}{2n}\operatorname{Tang}\frac{m\pi}{2n}\operatorname{Log}2n + \frac{\pi}{n}\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}\frac{im\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\}$$

$(m+n = \text{num. imp.})$

$$\int_1^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}}\left\{\operatorname{Log}\frac{n}{\pi}+\operatorname{Log}(\operatorname{Log}y)\right\}dy$$

$$= \frac{\pi}{2n}\operatorname{Tang}\frac{m\pi}{2n}\operatorname{Log}n + \frac{\pi}{n}\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin}\frac{im\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(1-\frac{i}{n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\}$$

$(m+n = \text{num. par})$

atque, quia est

$$\int_1^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}}dy = -\int_0^1 \frac{z^{\frac{m}{2n}-\frac{1}{2}}-z^{\frac{m}{2n}-\frac{1}{2}}}{1-z}dz = \frac{\pi}{2n}\operatorname{Tang}\frac{m\pi}{2n},$$

etiam

$$\left.\begin{aligned}&\int_1^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}}\operatorname{Log}(\operatorname{Log}y)\,dy\\ &= \frac{\pi}{2n}\operatorname{Tang}\frac{m\pi}{2n}\operatorname{Log}2\pi + \frac{\pi}{n}\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}\frac{im\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\}\end{aligned}\right\}\quad(14)$$

$(m+n = \text{num. imp.})$

$$\left.\begin{aligned}&\int_1^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}}\operatorname{Log}(\operatorname{Log}y)\,dy\\ &= \frac{\pi}{2n}\operatorname{Tang}\frac{m\pi}{2n}\operatorname{Log}\pi + \frac{\pi}{n}\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin}\frac{im\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(1-\frac{i}{n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\}\end{aligned}\right\}$$

$(m+n = \text{num. par})$

Hinc, si $m=1$, sequitur

$$\left.\begin{aligned}\int_1^\infty \frac{y^{n-2}\,\mathrm{Log}(\mathrm{Log}\,y)\,dy}{1+y^2+y^4\ldots+y^{2(n-1)}} &= \frac{\pi}{2n}\mathrm{Tang}\frac{\pi}{2n}\mathrm{Log}\,2\pi+\frac{\pi}{n}\sum_{i=1}^{i=n-1}(-1)^{i-1}\mathrm{Sin}\frac{i\pi}{n}\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\}\\ &\qquad (n=\text{num. par})\\ \int_1^\infty \frac{y^{n-2}\,\mathrm{Log}(\mathrm{Log}\,y)\,dy}{1+y^2+y^4\ldots+y^{2(n-1)}} &= \frac{\pi}{2n}\mathrm{Tang}\frac{\pi}{2n}\mathrm{Log}\,\pi+\frac{\pi}{n}\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Sin}\frac{i\pi}{n}\mathrm{Log}\left\{\frac{\Gamma\left(1-\frac{i}{n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\}\\ &\qquad (n=\text{num. imp.})\end{aligned}\right\}(15)$$

Ex. gr. posito in illa $n=2$ et in hac $n=3$, habebimus, si in hac y^2 in y mutatur,

$$\int_1^\infty \frac{\mathrm{Log}(\mathrm{Log}\,y)\,dy}{1+y^2}=\frac{\pi}{2}\mathrm{Log}\left\{\frac{(2\pi)^{\frac{1}{2}}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right\},$$

$$\int_1^\infty \frac{\mathrm{Log}(\mathrm{Log}\,y)\,dy}{1+y+y^2}=\frac{\pi}{\sqrt{3}}\mathrm{Log}\left\{\frac{(2\pi)^{\frac{1}{2}}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right\}.$$

§. 6. Ex antecedentibus jam apparet, transcendentem illam, quam per $L(a,x)$ designavimus, finite per Γ exprimi posse, quum a in ratione qualibet commensurabili ad π est. Nunc in sequentibus tribus §.§:is circa proprietates quasdam ejus maxime notandas ulterius versabimur. Sit igitur in priori formularum (12) $a=\frac{\pi}{2}$, erit

$$L\left(\tfrac{\pi}{2},x\right)=2\,\mathrm{Log}\left\{\frac{2\Gamma\left(\frac{x+3}{4}\right)}{\Gamma\left(\frac{x+1}{4}\right)}\right\},\quad\ldots\quad(16)$$

unde, si $x+2$ loco ipsius x ponitur, addendo invenimus

$$L\left(\tfrac{\pi}{2},x+2\right)+L\left(\tfrac{\pi}{2},x\right)=2\,\mathrm{Log}\,(x+1)\quad\ldots\quad(17)$$

Sin vero in (16) $2-x$ loco ipsius x substituimus, e relatione cognita

$$\Gamma(a)\,\Gamma(1-a)=\frac{\pi}{\mathrm{Sin}\,a\pi},$$

similiter addendo erit

$$L\left(\tfrac{\pi}{2},x\right)+L\left(\tfrac{\pi}{2},2-x\right)=2\mathrm{Log}\left((x-1)\,\mathrm{Cotang}\frac{(x-1)\pi}{4}\right).\quad(18)$$

id, quod pro $x=1$ formulam (13) reddit

$$L\left(\tfrac{\pi}{2},1\right)=\mathrm{Log}\left(\tfrac{4}{\pi}\right).\quad\ldots\quad(19)$$

E formulis (17) et (18) functionem $L\left(\frac{\pi}{2},x\right)$ pro quolibet ipsius x valore cognitam habemus, si modo per totam periodum ab $x=0$ ad $x=1$ cognita sit. Præterea formulæ (17) et (19) docent, $L\left(\frac{\pi}{2},x\right)$ pro $x=$ quolibet numero integro impari finite per Logarithmos et π exprimi posse.

Sit jam in priori (12) $a=\frac{2\pi}{5}$; tunc erit

$$L\left(\tfrac{2\pi}{5},x\right)=2\,\mathrm{Sin}\tfrac{\pi}{5}\,\mathrm{Log}\left\{\frac{6\Gamma\left(\frac{x+4}{6}\right)\Gamma\left(\frac{x+5}{6}\right)}{\Gamma\left(\frac{x+1}{6}\right)\Gamma\left(\frac{x+2}{6}\right)}\right\},$$

unde simili modo, ac supra, obtinebimus

$$\left.\begin{aligned} &L(\tfrac{2\pi}{3},x+3)+L(\tfrac{2\pi}{3},x)=2\operatorname{Sin}\tfrac{\pi}{3}\operatorname{Log}\big((x+2)(x+1)\big)\\ &L(\tfrac{2\pi}{3},x)+L(\tfrac{2\pi}{3},3-x)\\ &=2\operatorname{Sin}\tfrac{\pi}{3}\operatorname{Log}\Big((x-2)(x-1).\operatorname{Tang}\tfrac{(x+2)\pi}{6}.\operatorname{Tang}\tfrac{(x+1)\pi}{6}\Big)\end{aligned}\right\}\ldots(20)$$

atque e posteriori pro $x=\frac{3}{2}$

$$L(\tfrac{2\pi}{3},\tfrac{3}{2})=2\operatorname{Sin}\tfrac{\pi}{3}\operatorname{Log}\big(\tfrac{1}{2}\operatorname{Cot}\tfrac{\pi}{12}\big)\quad\ldots\quad(21)$$

E formulis (20) sequitur, ut functio $L(\frac{2\pi}{3},x)$ pro omnibus ipsius x valoribus cognita sit, si modo pro quolibet valore inter $x=0$ et $x=\frac{3}{2}$ cognitam habeamus; præterea prior harum formularum una cum (21) docet, $L(\frac{2\pi}{3},x)$ finite per Logarithmos et functiones circulares exprimi posse pro $x=\frac{(2i+1)3}{2}$, designante i numerum quemlibet integrum.

§. 7. At vero relationes longe generaliores invenire possumus, e quibus omnes, quales fuerunt præcedentes, tamquam casus speciales derivari possunt. Nam, ubi $a=\frac{m\pi}{n}$ et $m+n=$ numerus impar est, habemus etiam

$$\operatorname{Tang}\tfrac{1}{2}a=\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia,\quad\ldots\quad(22)$$

quare prior formularum (12) in hanc formam transformari potest

$$L(a,x)=2\sum_{i=1}^{i=n+1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}\ldots(23)$$

Substituamus jam hic $x+n$ pro x; tunc addendo habebimus

$$L(a,x+n)+L(a,x)=2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}(x+i)\ldots(24)$$
$$(m+n=\text{num. imp.})$$

Si porro in (23) $n-x$ loco x substituitur, prodit etiam addendo

$$L(a,x)+L(a,n-x)=$$
$$=2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{x+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}+2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(1+\frac{i-x}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{i-x}{2n}\right)}\right\}$$

sive, cum sit

$$\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(1+\frac{i-x}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{i-x}{2n}\right)}\right\}=\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}-\frac{x+i}{2n}\right)}{\Gamma\left(1-\frac{x+i}{2n}\right)}\right\}$$

etiam

$$L(a,x)+L(a,n-x)=2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{2n\,\Gamma\left(\frac{3}{2}-\frac{x+i}{2n}\right)\Gamma\left(\frac{1}{2}+\frac{x+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)\Gamma\left(1-\frac{x+i}{2n}\right)}\right\}.$$

Est autem

$$\Gamma\left(\tfrac{3}{2}-\frac{x+i}{2n}\right)\Gamma\left(\tfrac{1}{2}+\frac{x+i}{2n}\right)=\frac{n-x-i}{2n}\cdot\frac{\pi}{\operatorname{Cos}\frac{(x+i)\pi}{2n}},$$

$$\Gamma\left(\frac{x+i}{2n}\right)\Gamma\left(1-\frac{x+i}{2n}\right)=\frac{\pi}{\operatorname{Sin}\frac{(x+i)\pi}{2n}},$$

unde sequitur

$$L(a,x)+L(a,n-x)=2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}\left((n-x-i)\operatorname{Tang}\frac{(x+i)\pi}{2n}\right)$$

sive denique, posito $n-i$ loco i,

$$L(a,x)+L(a,n-x)=2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}\left((x-i)\operatorname{Cotang}\frac{(x-i)\pi}{2n}\right). \quad (25)$$

$(m+n=\text{num. imp.})$

Constat igitur e formulis (24) et (25), functionem $L(a,x)$ pro quovis ipsius x valore cognitam esse, si modo per totam periodum ab $x=o$ ad $x=\frac{n}{2}$ cognita sit, exsistente $m+n$ numero impari.

Posito in (25) $x=\frac{n}{2}$, habemus formulam

$$L(a,\tfrac{n}{2})=\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}\left((\tfrac{n}{2}-i)\operatorname{Cot}(\tfrac{\pi}{4}-\tfrac{i\pi}{2n})\right). \quad (26)$$

$(m+n=\text{num. imp.})$

quæ quidem pro $i=\frac{n}{2}$ expressionem Log $o.\infty$ præsentat; cujus tamen verus valor facile invenitur $\operatorname{Log}(\frac{2n}{\pi})$. Hæc formula una cum (25) docet, functionem $L(a,x)$, si $m+n$ numerus impar est, finite per Logarithmos et functiones circulares exprimi posse, posito $x=\frac{(2i+1)n}{2}$, designante i numerum quemvis integrum.

Consimili fere modo relationes analogas e posteriori formularum (12) derivare possumus. Nam substituatur ibi $x+n$ loco x, erit

$$L(a,x+n)=\operatorname{Tang}\tfrac{1}{2}a\operatorname{Log} n+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{2n+x-i}{n}\right)}{\Gamma\left(\frac{x+n+i}{n}\right)}\right\},$$

unde subtrahendo

$$L(a,x+n)-L(a,x)=2\sum_{i=1}^{i=n-1}{}^{2}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}\left\{\frac{x+n-i}{x+i}\right\}. \quad (27)$$

$(m+n=\text{num. par})$

Ceterum, si in eadem (12) $n-x$ pro x ponitur, fit

$$L(a,n-x)=\operatorname{Tang}\tfrac{1}{2}a\operatorname{Log} n+2\sum_{i=1}^{i=\frac{n-1}{2}}{}^{2}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{2n-x-i}{n}\right)}{\Gamma\left(\frac{n+i-x}{n}\right)}\right\},$$

unde, subtractione facta, e cognita functionis Γ ratione,

$$L(a,x)-L(a,n-x)=2\sum_{i=1}^{i=n-1}{}^{2}(-1)^{i-1}\operatorname{Sin} ia\operatorname{Log}\left\{\frac{x-i}{n-x-i}\cdot\frac{\operatorname{Sin}\frac{(x+i)\pi}{n}}{\operatorname{Sin}\frac{(x-i)\pi}{n}}\right\}. \quad (28)$$

$(m+n=\text{num. par})$

Formularum (27) et (28) beneficio sequitur, ut, exsistente $m+n$ numero pari, functionem $L(a,x)$ pro quovis ipsius x valore cognitam omnino habeamus, si modo per totam periodum ab $x=o$ ad $x=\frac{n}{2}$ cognita sit.

§. 8. Ponatur jam successive loco x

$$x,\ x+\tfrac{2n}{r},\ x+\tfrac{4n}{r},\ x+\tfrac{6n}{r},\ \ldots\ x+\tfrac{\overline{r-1}\cdot 2n}{r}$$

in priori formularum (12), et

$$x, x+\tfrac{n}{r}, x+\tfrac{2n}{r}, x+\tfrac{3n}{r} \ldots x+\tfrac{\overline{r-1}.n}{r}$$

in posteriori; habebimus

$$L(a,x) = \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} 2n + 2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\},$$

$$L(a,x+\tfrac{2n}{r}) = \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} 2n + 2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}+\frac{1}{r}\right)}{\Gamma\left(\frac{x+i}{2n}+\frac{1}{r}\right)}\right\},$$

. . . . . .

$$L(a,x+\tfrac{\overline{r-1}.2n}{r}) = \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} 2n + 2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}+\frac{r-1}{r}\right)}{\Gamma\left(\frac{x+i}{2n}+\frac{r-1}{r}\right)}\right\},$$

$(m+n=\text{num. imp.})$

et

$$L(a,x) = \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} n + 2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i}{n}\right)}{\Gamma\left(\frac{x+i}{n}\right)}\right\},$$

$$L(a,x+\tfrac{n}{r}) = \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} n + 2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i}{n}+\frac{1}{r}\right)}{\Gamma\left(\frac{x+i}{n}+\frac{1}{r}\right)}\right\},$$

. . . . . .

$$L(a,x+\tfrac{\overline{r-1}.n}{r}) = \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} n + 2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i}{n}+\frac{r-1}{r}\right)}{\Gamma\left(\frac{x+i}{n}+\frac{r-1}{r}\right)}\right\}.$$

$(m+n=\text{num. par})$

Cum autem in genere sit

$$\Gamma(y)\Gamma(y+\tfrac{1}{r})\Gamma(y+\tfrac{2}{r})\ldots\Gamma(y+\tfrac{r-1}{r}) = \Gamma(ry).(2\pi)^{\frac{r-1}{2}} r^{\frac{1}{2}-ry},$$

summando invenimus

$$\left.\begin{aligned} &L(a,x)+L(a,x+\tfrac{2n}{r})+L(a,x+\tfrac{4n}{r})\ldots+L(a,x+\tfrac{\overline{r-1}.2n}{r}) \\ &= r\operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} 2n + 2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{r}{2}+\frac{r(x+i)}{2n}\right)}{r^{\frac{r}{2}}\Gamma\left(\frac{r(x+i)}{2n}\right)}\right\} \\ &\qquad (m+n=\text{num. imp.}) \\ &L(a,x)+L(a,x+\tfrac{n}{r})+L(a,x+\tfrac{2n}{r})\ldots+L(a,x+\tfrac{\overline{r-1}.n}{r}) \\ &= r\operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} n + 2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(r+\frac{r(x-i)}{n}\right)}{r^{r-\frac{2ri}{n}}\Gamma\left(\frac{r(x+i)}{n}\right)}\right\} \\ &\qquad (m+n=\text{num. par}) \end{aligned}\right\} \ldots (29)$$

Facile apparet, priorem harum summarum per Logarithmos exprimi posse, cum r numerus par est. Ceterum formulæ (29) ita etiam exhiberi possunt, ut, positis

$$f'(u) = (u^2+x^2)(u^2+x+\tfrac{2n}{r})^2(u^2+x+\tfrac{4n}{r})^2\ldots(u^2+x+\tfrac{\overline{r-1}.2n}{r})^2$$

$$f''(u) = (u^2+x^2)(u^2+x+\tfrac{n}{r})^2(u^2+x+\tfrac{2n}{r})^2\ldots(u^2+x+\tfrac{\overline{r-1}.n}{r})^2$$

sit

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} \operatorname{Log}(f'(u))\,du$$

$$= r\operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} 2n + 2\sum_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{r}{2}+\frac{r(x+i)}{2n}\right)}{r^{\frac{r}{2}}\Gamma\left(\frac{r(x+i)}{2n}\right)}\right\},$$

$(m+n=\text{num. imp.})$

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} \operatorname{Log}(f''(u))\,du$$

$$= r \operatorname{Tang}\tfrac{1}{2}a \operatorname{Log} n + 2\sum_{i=1}^{i=\frac{n-1}{2}} (-1)^{i-1} \operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(r+\frac{r(x-i)}{n}\right)}{r^{r-\frac{2ri}{n}}\Gamma\left(\frac{r(x+i)}{n}\right)}\right\}.$$

$(m+n = \text{num. par})$

§. 9. Tribuamus jam functioni $\varphi(x)$ aliam formam, ex. gr.

$$\varphi(x) = \frac{1}{x^s};$$

tunc erit, posito $r = 1$,

$$\psi(u) = \tfrac{1}{2} \cdot \frac{(x+u\sqrt{-1})^s + (x-u\sqrt{-1})^s}{(x^2+u^2)^s},$$

atque ex formula (6) accipies

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} \cdot \frac{(x+u\sqrt{-1})^s + (x-u\sqrt{-1})^s}{(x^2+u^2)^s}\,du = 2\sum_{i=1}^{i=\infty} (-1)^{i-1} \frac{\operatorname{Sin} ia}{(x+i)^s}. \quad (30)$$

Est vero cognita relatio

$$\frac{\operatorname{Sin} a}{1+2y\operatorname{Cos} a+y^2} = \sum_{i=1}^{i=\infty} (-1)^{i-1} y^{i-1} \operatorname{Sin} ia,$$

quæ, si utrimque multiplicetur per $y^x \left(\operatorname{Log}\frac{1}{y}\right)^{s-1} dy$, integratione inter $y = 0$ et $y = 1$ instituta, dabit

$$\frac{\operatorname{Sin} a}{\Gamma(s)} \int_0^1 \frac{y^x \left(\operatorname{Log}\frac{1}{y}\right)^{s-1} dy}{1+2y\operatorname{Cos} a+y^2} = \sum_{i=1}^{i=\infty} (-1)^{i-1} \frac{\operatorname{Sin} ia}{(x+i)^s},$$

SPECIMEN ANALYTICUM,

THEOREMATA QUÆDAM NOVA

DE INTEGRALIBUS DEFINITIS, SUMMATIONE SERIERUM
EARUMQUE IN ALIAS SERIES TRANSFORMATIONE
EXHIBENS,

CUJUS PARTEM TERTIAM

VENIA AMPL. FAC. PHILOS. UPSAL.

P. P.

MAG. **CAROLUS JOHANNES MALMSTEN**
MATH. INFER. DOCENS

ET

JOHANNES ALFRED GRENANDER
Vestrog.

IN AUDITORIO GUSTAV. DIE IX APR. MDCCCXLII.

H. A. M. S.

UPSALIAE,
EXCUDEBANT REGIAE ACADEMIAE TYPOGRAPHI.

cum sit in genere

$$\int_0^1 y^n \left(\operatorname{Log}\frac{1}{y}\right)^{s-1} dy = \frac{\Gamma(s)}{(n+1)^s}.$$

Hinc, comparatione cum (30) facta, prodit

$$\left.\begin{aligned}\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} \cdot \frac{(x+u\sqrt{-1})^s+(x-u\sqrt{-1})^s}{(x^2+u^2)^s}\,du \\ = \frac{2\operatorname{Sin} a}{\Gamma(s)} \cdot \int_0^1 \frac{y^x \left(\operatorname{Log}\frac{1}{y}\right)^{s-1} dy}{1+2y\operatorname{Cos} a+y^2}.\end{aligned}\right\} \ldots (31)$$

§. 10. Sit jam $s > o$ sed < 1, et ponamus $x = o$; tunc, quia

$$(u\sqrt{-1})^s + (-u\sqrt{-1})^s = 2u^s \operatorname{Cos}\frac{s\pi}{2},$$

obtinemus

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} \cdot \frac{du}{u^s} = \frac{\operatorname{Sin} a}{\operatorname{Cos}\frac{s\pi}{2}\,\Gamma(s)} \cdot \int_0^1 \frac{dy}{1+2y\operatorname{Cos} a+y^2} \cdot \frac{1}{\left(\operatorname{Log}\frac{1}{y}\right)^{1-s}}.$$

Hinc porro, si $a = \frac{m\pi}{n}$ et $e^{-\frac{\pi u}{n}} = y$ ponitur, unde fit $u = \frac{n}{\pi}\operatorname{Log}\frac{1}{y}$, $du = -\frac{n}{\pi}\frac{dy}{y}$, transformatione facta, eruitur

$$\left.\begin{aligned}\int_0^1 \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}} \cdot \frac{dy}{\left(\operatorname{Log}\frac{1}{y}\right)^s} = \\ = \frac{\left(\frac{\pi}{n}\right)^{1-s} \cdot \operatorname{Sin}\frac{m\pi}{n}}{\operatorname{Cos}\frac{s\pi}{2}\,\Gamma(s)} \cdot \int_0^1 \frac{dy}{1+2y\operatorname{Cos}\frac{m\pi}{n}+y^2} \cdot \frac{1}{\left(\operatorname{Log}\frac{1}{y}\right)^{1-s}}\end{aligned}\right\} \ldots (32)$$

et pro $m=1$

$$\left.\begin{aligned}\int_0^1 \frac{y^{n-2}\,dy}{1+y^2+y^4..+y^{2(n-1)}}\cdot\frac{1}{\left(\operatorname{Log}\frac{1}{y}\right)^s} &= \\ = \frac{\left(\frac{\pi}{n}\right)^{1-s}\cdot\operatorname{Sin}\frac{\pi}{n}}{\operatorname{Cos}\frac{s\pi}{2}\,\Gamma(s)}\cdot\int_0^1 \frac{dy}{1+2y\operatorname{Cos}\frac{\pi}{n}+y^2}\cdot\frac{1}{\left(\operatorname{Log}\frac{1}{y}\right)^{1-s}}&.\end{aligned}\right\}..(33)$$

Appellemus

$$G(s)=\int_0^1 \frac{\left(\operatorname{Log}\frac{1}{y}\right)^{s-1}}{1+y+y^2}\,dy,$$

$$\mathfrak{G}(s)=\int_0^1 \frac{\left(\operatorname{Log}\frac{1}{y}\right)^{s-1}}{1+y^2}\,dy.$$

Posito $n=3$, et mutato y^2 in y, formula (33) dabit

$$G(1-s)=\frac{\left(\frac{2\pi}{3}\right)^{1-s}\operatorname{Sin}\frac{\pi}{3}}{\operatorname{Cos}\frac{s\pi}{2}\,\Gamma(s)}G(s), \qquad (34)$$

atque ex eâdem pro $n=2$ immediate colligitur

$$\mathfrak{G}(1-s)=\frac{\left(\frac{\pi}{2}\right)^{1-s}}{\operatorname{Cos}\frac{s\pi}{2}\,\Gamma(s)}\mathfrak{G}(s). \qquad (35)$$

Ecce simplices et notandæ equationes, quæ functiones $G(s)$, $\mathfrak{G}(s)$ earumque complementarias $G(1-s)$, $\mathfrak{G}(1-s)$ inter se

conjungunt; in quo respectu analogæ omnino sunt cum hac cognita functionis Γ formula

$$\Gamma(a)\,\Gamma(1-a)=\frac{\pi}{\operatorname{Sin} a\pi}.$$

Ceterum, quia sunt

$$\frac{1}{1+y+y^2}=1-y+y^3-y^4+y^6-\text{etc.}$$

$$\frac{1}{1+y^2}=1-y^2+y^4-y^6+\text{etc.}$$

constat etiam

$$G(1-s)=\Gamma(1-s)\left\{1-\frac{1}{2^{1-s}}+\frac{1}{4^{1-s}}-\frac{1}{5^{1-s}}+\frac{1}{7^{1-s}}-\text{etc.}\right\},$$

$$\mathfrak{G}(1-s)=\Gamma(1-s)\left\{1-\frac{1}{3^{1-s}}+\frac{1}{5^{1-s}}-\frac{1}{7^{1-s}}+\frac{1}{9^{1-s}}-\text{etc.}\right\},$$

unde formulæ (34) et (35) has relationes suppeditant

$$\left.\begin{aligned}&1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\frac{1}{7^s}-\frac{1}{8^s}+\text{etc.}\\ &=\frac{\operatorname{Cos}\frac{s\pi}{2}\,\Gamma(1-s)}{\left(\frac{2\pi}{3}\right)^{1-s}\operatorname{Sin}\frac{\pi}{3}}\left\{1-\frac{1}{2^{1-s}}+\frac{1}{4^{1-s}}-\frac{1}{5^{1-s}}+\frac{1}{7^{1-s}}-\text{etc.}\right\}\\ &1-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}-\frac{1}{11^s}+\text{etc.}\\ &=\frac{\operatorname{Cos}\frac{s\pi}{2}\,\Gamma(1-s)}{\left(\frac{\pi}{2}\right)^{1-s}}\left\{1-\frac{1}{3^{1-s}}+\frac{1}{5^{1-s}}-\frac{1}{7^{1-s}}+\text{etc.}\right\}\end{aligned}\right\}...(36)$$

Ex his vero formulis, respectu s differentiando, alias etiam novas habebimus. Inde enim, cum sit

$$1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}-\text{etc.}=\frac{\pi}{3\sqrt{3}};\quad 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\text{etc.}=\tfrac{\pi}{4}$$

et

$$\frac{d\Gamma(1-s)}{ds}=-\Gamma(1-s).Z'(1-s),$$

(posito ex notatione *Cel:i Legendre* $Z'(a)=\dfrac{d\operatorname{Log}\Gamma(a)}{da}$), si, differentiatione facta, $s=0$ ponitur, obtinebimus

$$-\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin}\tfrac{i\pi}{3}\operatorname{Log}i=\frac{1}{2\sqrt{3}}\left(C+\operatorname{Log}\frac{2\pi}{3}\right)+\frac{3}{2\pi}\left\{\frac{\operatorname{Log}1}{1}-\frac{\operatorname{Log}2}{2}+\frac{\operatorname{Log}4}{4}-\text{etc.}\right\},$$

$$-\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin}\tfrac{i\pi}{2}\operatorname{Log}i=\tfrac{1}{2}C+\tfrac{1}{2}\operatorname{Log}\frac{\pi}{2}+\frac{2}{\pi}\left\{\frac{\operatorname{Log}1}{1}-\frac{\operatorname{Log}3}{3}+\frac{\operatorname{Log}5}{5}-\text{etc.}\right\},$$

exsistente $C=-Z'(1)$ cognita illa Euleri constante 0,577216...

Habemus autem e formulis (7) et (12)

$$2\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin}\tfrac{i\pi}{3}\operatorname{Log}i=L(\tfrac{\pi}{3},0)=-\sqrt{3}\operatorname{Log}\left\{\frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{2}}\Gamma(\frac{2}{3})}\right\},$$

$$2\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin}\tfrac{i\pi}{2}\operatorname{Log}i=L(\tfrac{\pi}{2},0)=-2\operatorname{Log}\left\{\frac{\Gamma(\frac{1}{4})}{2\Gamma(\frac{3}{4})}\right\}=-2\operatorname{Log}\left\{\frac{\pi}{2^{\frac{1}{2}}\left(\Gamma(\frac{3}{4})\right)^2}\right\}$$

unde denique, calculo facto, invenimus

$$\left.\begin{aligned}\frac{\operatorname{Log}1}{1}-\frac{\operatorname{Log}2}{2}+\frac{\operatorname{Log}4}{4}-\frac{\operatorname{Log}5}{5}+\text{etc.}&=\frac{\pi}{\sqrt{3}}\operatorname{Log}\left\{\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\right\}-\frac{\pi}{3\sqrt{3}}\left(C+\operatorname{Log}2\pi\right)\\ \frac{\operatorname{Log}1}{1}-\frac{\operatorname{Log}3}{5}+\frac{\operatorname{Log}5}{5}-\frac{\operatorname{Log}7}{7}+\text{etc.}&=\frac{\pi}{4}\left(\operatorname{Log}\pi-C\right)-\pi\operatorname{Log}\Gamma(\tfrac{3}{4})\end{aligned}\right\}(37)$$

et

$$\left.\begin{aligned}\frac{1.4^{\frac{1}{4}}.7^{\frac{1}{7}}.10^{\frac{1}{10}}\cdots}{2^{\frac{1}{2}}.5^{\frac{1}{5}}.8^{\frac{1}{8}}.11^{\frac{1}{11}}\cdots}&=\left\{\frac{e^{-\frac{1}{3}C}.\Gamma(\frac{1}{3})}{(2\pi)^{\frac{1}{3}}\Gamma(\frac{2}{3})}\right\}^{\frac{\pi}{\sqrt{3}}}\\ \frac{1^{\frac{1}{1}}.5^{\frac{1}{5}}.9^{\frac{1}{9}}.13^{\frac{1}{13}}\cdots}{3^{\frac{1}{3}}.7^{\frac{1}{7}}.11^{\frac{1}{11}}.15^{\frac{1}{15}}\cdots}&=\left\{\frac{\pi e^{-C}}{\left(\Gamma(\frac{3}{4})\right)^4}\right\}^{\frac{\pi}{4}}\end{aligned}\right\}\quad\ldots\ldots(38)$$

§. 11. Posito

$$t=\operatorname{Arc\,Tang}\frac{u}{x},$$

cognitum utique est

$$\frac{(x+u\sqrt{-1})^s+(x-u\sqrt{-1})^s}{(x^2+u^2)^s}=\frac{2\operatorname{Cos}st}{(x^2+u^2)^{\frac{s}{2}}},$$

unde e formula (31) prodit

$$\int_0^\infty\frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}.\frac{\operatorname{Cos}st\,du}{(x^2+u^2)^{\frac{s}{2}}}=\frac{\operatorname{Sin}a}{\Gamma(s)}\int_0^1\frac{y^x(\operatorname{Log}\frac{1}{y})^{s-1}dy}{1+2y\operatorname{Cos}a+y^2}$$

et pro $s=1$

$$\int_0^\infty\frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}.\frac{du}{x^2+u^2}=\frac{\operatorname{Sin}a}{x}\int_0^1\frac{y^x dy}{1+2y\operatorname{Cos}a+y^2}\ldots(39)$$

Quoniam vero, ex demonstratione *Cel:i Legendre* [4]), integrale

$$\int_0^1\frac{y^x\,dy}{1+2y\operatorname{Cos}a+y^2}$$

per functionem $Z'(x)=\dfrac{d\operatorname{Log}\Gamma(x)}{dx}$ semper exprimi potest ex-

[4]) Vide *Legendre Exerc. de Calc. Int.* Tom, II. pag, 162 seqq.

sistente a in ratione qualibet commensurabili ad π, quicumque ipsi x valor tribuatur, atque præterea patet, posito x rationali, idem integrale, quodcumque sit a, semper inveniri posse; sequitur inde, ut in utroque hoc casu valor integralis

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} \cdot \frac{du}{x^2+u^2}$$

cognitus etiam habeatur.

Jam vero, differentiando respectu a formulam (39), novas etiam inde obtinebimus. Ad quod tamen cum necessarium sit relationes quasdam noscere, quas non antea habemus cognitas, e re fore haud dubitavimus, theoremata quædam, ad quæ nobis contigerit pervenire, quæque attentione Mathematicorum omnino digna videantur, ante omnia hoc loco proponere.

§. 12. Posito igitur

$$P=\frac{\operatorname{Sin} a}{1+2y\operatorname{Cos} a+y^2} \quad \text{et} \quad Q=\frac{y+\operatorname{Cos} a}{1+2y\operatorname{Cos} a+y^2};$$

inquiramus in generalem expressionem ipsorum $\frac{d^rP}{da^r}$ et $\frac{d^rQ}{da^r}$.

Sit brevitatis caussa

$$i=\sqrt{-1}; \quad Z_0=(1+ye^{-ai})^{-1}=e^{ai}(y+e^{ai})^{-1},$$

$$'Z_0=(1+ye^{ai})^{-1}=e^{-ai}(y+e^{-ai})^{-1},$$

$$Z_r=\frac{d^r\left(e^{ai}(y+e^{ai})^{-1}\right)}{da^r}; \quad 'Z_r=\frac{d^r\left(e^{-ai}(y+e^{-ai})^{-1}\right)}{da^r};$$

unde facile colligitur,

$$\left.\begin{aligned}\frac{Z_r+'Z_r}{2}&=-y\frac{d^rQ}{da^r}\\ \frac{Z_r-'Z_r}{2\sqrt{-1}}&=y\frac{d^rP}{da^r}.\end{aligned}\right\} \quad \ldots\ldots (\alpha)$$

Ceterum in promtu est, invento Z_r, mutato tantum ipsius a signo, exinde $'Z_r$ inveniri. Cum jam in genere sit

$$Z_{r+1}=\frac{dZ_r}{da}$$

et

$$\frac{dZ_0}{da}=i(Z_0-Z_0^2) \quad \ldots\ldots (\beta)$$

facile successive habebimus

$$\frac{Z_1}{i}=Z_0-Z_0^2; \quad \frac{Z_2}{i^2}=Z_0-3Z_0^2+1.2.Z_0^3,$$

$$\frac{Z_3}{i^3}=Z_0-7Z_0^2+12Z_0^3-1.2.3.Z_0^4,$$

atque in genere

$$\frac{Z_r}{i^r}=\sum_{k=1}^{k=r+1}(-1)^{k-1}\,\mathfrak{a}^{(r,k)}\,Z_0^k, \quad \ldots\ldots (\gamma)$$

designante $\mathfrak{a}^{(r,k)}$ generalem ipsius Z_0^k coëfficientem in Z_r, de qua tantum novimus

$$\mathfrak{a}^{(r,o)}=0; \quad \mathfrak{a}^{(r,r+1)}=\Gamma(r+1), \quad \text{et} \quad \mathfrak{a}^{(r,r+m)}=0, \text{ ubi } m>1.$$

Mutatis r in $r+1$ et k in $k+1$, fit

$$\frac{Z_{r+1}}{i^{r+1}} = \sum_{k=0}^{k=r+1} (-1)^k \, {}^{(r+1,k+1)}\mathfrak{a} \; Z_0^{k+1} \quad . \quad . \quad . \quad . \quad (\delta)$$

Jam porro, si formula (γ) respectu a differentiatur, ope formulæ (β) habebimus

$$\frac{Z_{r+1}}{i^{r+1}} = \sum_{k=1}^{k=r+1} (-1)^{k-1} k \, {}^{(r,k)}\mathfrak{a} \; Z_0^{k} + \sum_{k=0}^{k=r+1} (-1)^k k \, {}^{(r,k)}\mathfrak{a} \; Z_0^{k+1}$$

seu, posito in priori summa $k+1$ loco k,

$$\frac{Z_{r+1}}{i^{r+1}} = \sum_{k=0}^{k=r+1} (-1)^k \left\{ (k+1) \, {}^{(r,k+1)}\mathfrak{a} + k \, {}^{(r,k)}\mathfrak{a} \right\} Z_0^{k+1},$$

unde, comparatione cum (δ) facta, hanc functionis ${}^{(r,k)}\mathfrak{a}$ determinationem obtinemus

$${}^{(r+1,k+1)}\mathfrak{a} = (k+1) \, {}^{(r,k+1)}\mathfrak{a} + k \, {}^{(r,k)}\mathfrak{a} \quad . \quad . \quad . \quad (\varepsilon)$$

Ut vero *Clausen* in Diario *Cel:i Crelle* monstravit, ejusmodi æquationi satisfiet ponendo

$${}^{(r,k)}\mathfrak{a} = k^r - \frac{k-1}{1}(k-1)^r + \frac{k-1}{1}\cdot\frac{k-2}{2}(k-2)^r - \frac{k-1}{1}\cdot\frac{k-2}{2}\cdot\frac{k-3}{3}(k-3)^r + \text{etc.}, \quad (\zeta)$$

unde igitur ${}^{(r,k)}\mathfrak{a}$ omnino cognitum habemus. Ceterum patet, exsistentibus r et k numeris integris, hanc seriem semper e finito terminorum numero constare.

Determinato jam Z_r per formulam (γ), exinde ${}'Z_r$ facile deducetur. Nam, mutato tantum ipsius a signo, formula (γ) dabit

$$\frac{{}'Z_r}{i^r} = (-1)^r \sum_{k=1}^{k=r+1} (-1)^{k-1} \, {}^{(r,k)}\mathfrak{a} \; {}'Z_0^{k} \quad . \quad . \quad . \quad . \quad (\eta)$$

unde addendo et subtrahendo accipies

$$\frac{Z_r + {}'Z_r}{i^r} = \sum_{k=1}^{k=r+1} (-1)^{k-1} \, {}^{(r,k)}\mathfrak{a} \left\{ Z_0^k + (-1)^r \, {}'Z_0^k \right\},$$

$$\frac{Z_r - {}'Z_r}{i^r} = \sum_{k=1}^{k=r+1} (-1)^{k-1} \, {}^{(r,k)}\mathfrak{a} \left\{ Z_0^k - (-1)^r \, {}'Z_0^k \right\}.$$

Ex his vero, cum formulis (α) comparatis, prout $r = 2n+1$ aut $r = 2n$ sit, habebimus

$$\left.\begin{aligned}
y\frac{d^{2n+1}P}{da^{2n+1}} &= (-1)^n \sum_{k=1}^{k=2(n+1)} (-1)^{k-1} \, {}^{(2n+1,k)}\mathfrak{a} \cdot \left\{ \frac{Z_0^k + {}'Z_0^k}{2} \right\} \\
y\frac{d^{2n}P}{da^{2n}} &= (-1)^n \sum_{k=1}^{k=2n+1} (-1)^{k-1} \, {}^{(2n,k)}\mathfrak{a} \cdot \left\{ \frac{Z_0^k - {}'Z_0^k}{2\sqrt{-1}} \right\} \\
y\frac{d^{2n+1}Q}{da^{2n+1}} &= (-1)^n \sum_{k=1}^{k=2(n+1)} (-1)^{k-1} \, {}^{(2n+1,k)}\mathfrak{a} \cdot \left\{ \frac{Z_0^k - {}'Z_0^k}{2\sqrt{-1}} \right\} \\
y\frac{d^{2n}Q}{da^{2n}} &= (-1)^{n-1} \sum_{k=1}^{k=2n+1} (-1)^{k-1} \, {}^{(2n,k)}\mathfrak{a} \cdot \left\{ \frac{Z_0^k + {}'Z_0^k}{2} \right\}
\end{aligned}\right\}_g \quad . \quad (\theta)$$

Statuamus vero in Z_0 et ${}'Z_0$

$$1 + y\,\text{Cos}\,a = \beta\,\text{Cos}\,\psi, \qquad y\,\text{Sin}\,a = \beta\,\text{Sin}\,\psi,$$

unde, eliminatione facta,

$$\psi = \text{Arc Tang}\,\frac{y\,\text{Sin}\,a}{1 + y\,\text{Cos}\,a}; \quad \beta^2 = 1 + 2y\,\text{Cos}\,a + y^2;$$

tunc, calculo effecto, habebimus

$$\frac{Z_0^k + {}'Z_0^k}{2} = \frac{\text{Cos}\,k\psi}{\beta^k} \quad \text{et} \quad \frac{Z_0^k - {}'Z_0^k}{2\sqrt{-1}} = \frac{\text{Sin}\,k\psi}{\beta^k},$$

atque idcirco, substitutis ipsorum ψ et β valoribus

$$\left.\begin{aligned}
\frac{d^{2n+1}P}{da^{2n+1}} &= \frac{(-1)^n}{y}\cdot\sum_{k=1}^{k=2(n+1)_{k-1}}(-1)\cdot\frac{{}^{(2n+1,k)}a\,\mathrm{Cos}\,k\left(\mathrm{Arc\,Tang}\frac{y\,\mathrm{Sin}\,a}{1+y\,\mathrm{Cos}\,a}\right)}{k\,(1+2y\,\mathrm{Cos}\,a+y^2)^{\frac{k}{2}}}\\
\frac{d^{2n}P}{da^{2n}} &= \frac{(-1)^n}{y}\cdot\sum_{k=1}^{k=2n+1_{k-1}}(-1)\cdot\frac{{}^{(2n,k)}a\,\mathrm{Sin}\,k\left(\mathrm{Arc\,Tang}\frac{y\,\mathrm{Sin}\,a}{1+y\,\mathrm{Cos}\,a}\right)}{k\,(1+2y\,\mathrm{Cos}\,a+y^2)^{\frac{k}{2}}}\\
\frac{d^{2n+1}Q}{da^{2n+1}} &= \frac{(-1)^n}{y}\cdot\sum_{k=1}^{k=2(n+1)_{k-1}}(-1)\cdot\frac{{}^{(2n+1,k)}a\,\mathrm{Sin}\,k\left(\mathrm{Arc\,Tang}\frac{y\,\mathrm{Sin}\,a}{1+y\,\mathrm{Cos}\,a}\right)}{k\,(1+2y\,\mathrm{Cos}\,a+y^2)^{\frac{k}{2}}}\\
\frac{d^{2n}Q}{da^{2n}} &= \frac{(-1)^{n-1}}{y}\cdot\sum_{k=1}^{k=2n+1_{k-1}}(-1)\cdot\frac{{}^{(2n,k)}a\,\mathrm{Cos}\,k\left(\mathrm{Arc\,Tang}\frac{y\,\mathrm{Sin}\,a}{1+y\,\mathrm{Cos}\,a}\right)}{k\,(1+2y\,\mathrm{Cos}\,a+y^2)^{\frac{k}{2}}}
\end{aligned}\right\}\ldots(\iota)$$

determinato jam ${}^{(r,k)}a$ per formulam (ζ). Quas ita nobis contigerit proponere generales ipsorum $\frac{d^rP}{da^r}$ et $\frac{d^rQ}{da^r}$ expressiones, eas multis in occasionibus usui fore credimus, præsertim cum iis accedant novæ relationes, quas paullo infra in formula (π) demonstrabimus.

Inventis jam formulis (ι), complures inde ipsius ${}^{(r,k)}a$ proprietates, e quibus duæ in sequentibus maxime nobis erunt necessariæ, facile erui possunt. Posito igitur $y=0$, unde

$$P=\mathrm{Sin}\,a,\qquad Q=\mathrm{Cos}\,a,$$

cum sit

$$\frac{d^{2n}P}{da^{2n}}=(-1)^n\,\mathrm{Sin}\,a,\qquad \frac{d^{2n+1}Q}{da^{2n+1}}=(-1)^{n+1}\,\mathrm{Sin}\,a,$$

SPECIMEN ANALYTICUM,

THEOREMATA QUÆDAM NOVA

DE INTEGRALIBUS DEFINITIS, SUMMATIONE SERIERUM
EARUMQUE IN ALIAS SERIES TRANSFORMATIONE
EXHIBENS,

CUJUS PARTEM QUARTAM

VENIA AMPL. FAC. PHILOS. UPSAL.

P. P.

MAG. CAROLUS JOHANNES MALMSTEN
MATH. INFER. DOCENS

ET

GEORG. ENHARD. OSSBAHR
Holmiens. Stip. Faltzburg.

IN AUDITORIO GUSTAV. DIE IX APR. MDCCCXLII.

H. P. M. S.

UPSALIAE,
EXCUDEBANT REGIAE ACADEMIAE TYPOGRAPHI.

has formulas habebimus

$$\left.\begin{array}{ll} \sum_{k=1}^{k=2(n+1)} (-1)^{k-1} \mathfrak{a}^{(2n+1,k)} = 0; & \sum_{k=1}^{k=2n+1} (-1)^{k-1} k \mathfrak{a}^{(2n,k)} = 1 \\ \sum_{k=1}^{k=2(n+1)} (-1)^{k-1} k \mathfrak{a}^{(2n+1,k)} = -1; & \sum_{k=1}^{k=2n+1} (-1)^{k-1} \mathfrak{a}^{(2n,k)} = 0 \end{array}\right\} \ldots (\varkappa)$$

quarum prima et quarta, e ratione

$$\mathfrak{a}^{(2n+1,2n+2)} = \Gamma(2n+2), \quad \mathfrak{a}^{(2n,2n+1)} = \Gamma(2n+1),$$

ita etiam possunt exhiberi

$$\sum_{k=1}^{k=n+1} (-1)^{k-1} \mathfrak{a}^{(2n+1,k)} = \Gamma(2n+2); \quad \sum_{k=1}^{k=2n} (-1)^{k-1} \mathfrak{a}^{(2n,k)} = -\Gamma(2n+1). \quad (\lambda)$$

Sit porro $y = 1$, unde $P = \frac{1}{2}\mathrm{Tang}\frac{a}{2}$, $Q = \frac{1}{2}$, et deinde $y = -1$, unde $P = \frac{1}{2}\mathrm{Cotang}\frac{a}{2}$, $Q = -\frac{1}{2}$; tunc, mutato a in $2a$, e prima et secunda formularum (ι) habebimus

$$\left.\begin{array}{l} \frac{d^{2n+1}(\mathrm{Tang}\, a)}{da^{2n+1}} = (-1)^n 2^{2(n+1)} \cdot \sum_{k=1}^{k=2(n+1)} (-1)^{k-1} \cdot \frac{\mathfrak{a}^{(2n+1,k)} \mathrm{Cos}\, ka}{2^k (\mathrm{Cos}\, a)^k} \\ \frac{d^{2n}(\mathrm{Tang}\, a)}{da^{2n}} = (-1)^n 2^{2n+1} \cdot \sum_{k=1}^{k=2n+1} (-1)^{k-1} \cdot \frac{\mathfrak{a}^{(2n,k)} \mathrm{Sin}\, ka}{2^k (\mathrm{Cos}\, a)^k} \\ \frac{d^{2n+1}(\mathrm{Cotang}\, a)}{da^{2n+1}} = -(-1)^n 2^{2(n+1)} \cdot \sum_{k=1}^{k=2(n+1)} (-1)^{k-1} \cdot \frac{\mathfrak{a}^{(2n+1,k)} \mathrm{Cos}\, k(\frac{1}{2}\pi - a)}{2^k (\mathrm{Sin}\, a)^k} \\ \frac{d^{2n}(\mathrm{Cotang}\, a)}{da^{2n}} = (-1)^n 2^{2n+1} \cdot \sum_{k=1}^{k=2n+1} (-1)^{k-1} \cdot \frac{\mathfrak{a}^{(2n,k)} \mathrm{Sin}\, k(\frac{1}{2}\pi - a)}{2^k (\mathrm{Sin}\, a)^k} \end{array}\right\} (\mu)$$

atque e tertia et quarta

$$\left.\begin{aligned} \sum_{k=1}^{k=2(n+1)} (-1)^{k-1} \cdot \frac{\overset{(2n+1,k)}{a}\ \mathrm{Sin}\, ka}{2^k (\mathrm{Cos}\, a)^k} &= 0 \\ \sum_{k=1}^{k=2n+1} (-1)^{k-1} \cdot \frac{\overset{(2n,k)}{a}\ \mathrm{Cos}\, ka}{2^k (\mathrm{Cos}\, a)^k} &= 0 \end{aligned}\right\} \ldots\ldots\ldots (\nu)$$

quarum postrema, pro $a=0$, dabit

$$\sum_{k=1}^{k=2n+1} (-1)^{k-1} \cdot \frac{\overset{(2n,k)}{a}}{2^k} = 0 \ldots\ldots\ldots (\xi)$$

His jam cognitis, in novas ipsorum $\frac{d^r P}{da^r}$, $\frac{d^r Q}{da^r}$ relationes inquiramus. Posito

$$M_r = \frac{yd(yd(\ldots yd(yQ)}{dy^r} \quad \text{et} \quad N_r = \frac{yd(yd(\ldots yd(yP)}{dy^r}, \ldots (o)$$

demonstrabimus, quæcumque sint a et y,

$$\left.\begin{aligned} y\frac{d^{2n+1}P}{da^{2n+1}} &= (-1)^n M_{2n+1}\,; & y\frac{d^{2n}P}{da^{2n}} &= (-1)^n N_{2n} \\ y\frac{d^{2n+1}Q}{da^{2n+1}} &= (-1)^{n+1} N_{2n+1}\,; & y\frac{d^{2n}Q}{da^{2n}} &= (-1)^n M_{2n} \end{aligned}\right\} \ldots (\pi)$$

Sit igitur

$$\mathcal{J}_r = \frac{yd(yd(\ldots yd(y(y+e^{-ai})^{-1}}{dy^r}\,; \quad {}'\mathcal{J}_r = \frac{yd(yd(\ldots yd(y(y+e^{ai})^{-1}}{dy^r},$$

unde sequitur

$$\left.\begin{aligned} \mathcal{J}_r + {}'\mathcal{J}_r &= 2M_r \\ \mathcal{J}_r - {}'\mathcal{J}_r &= 2\sqrt{-1}\, N_r \end{aligned}\right\} \ldots\ldots\ldots (\varrho)$$

Ceterum patet, cognito $\mathcal{J}_r$ exinde ${}'\mathcal{J}_r$ inveniri, mutato tantum signo ipsius a. Jam vero in promtu est

$$\mathcal{J}_0 = y(y+e^{-ai})^{-1} \text{ et in genere } \mathcal{J}_{r+1} = y\frac{d\mathcal{J}_r}{dy},$$

seu, posito $y+e^{-ai}=v$, unde $dy=dv$,

$$\mathcal{J}_0 = 1 - \frac{e^{-ai}}{v}, \text{ et } \mathcal{J}_{r+1} = (v-e^{-ai})\frac{d\mathcal{J}_r}{dv} \ldots\ldots (\sigma)$$

Habebimus igitur successive

$$\mathcal{J}_1 = \frac{e^{-ai}}{v} - \frac{e^{-2ai}}{v^2}\,; \quad \mathcal{J}_2 = -\frac{e^{-ai}}{v} + \frac{3e^{-2ai}}{v^2} - \frac{1.2.e^{-3ai}}{v^3},$$

atque in genere

$$\mathcal{J}_r = (-1)^{r-1} \sum_{k=1}^{k=r+1} (-1)^{k-1} \cdot \frac{\overset{(r,k)}{c}\ e^{-kai}}{v^k}, \ldots\ldots\ldots (\tau)$$

designante $\overset{(r,k)}{c}$ generalem ipsius $\frac{e^{-kai}}{v^k}$ coëfficientem in $\mathcal{J}_r$, de qua tantum novimus

$$\overset{(r,o)}{c} = 0\,; \quad \overset{(r,r+1)}{c} = \Gamma(r+1), \text{ et } \overset{(r,r+m)}{c} = 0, \text{ ubi } m > 1.$$

Positis jam $r+1$ et $k+1$ loco r et k, fit

$$\mathcal{J}_{r+1} = (-1)^r \sum_{k=o}^{k=r+1} (-1)^k \cdot \frac{\overset{(r+1,k+1)}{c}\ e^{-(k+1)ai}}{v^{k+1}}.$$

Formula vero (τ), respectu v differentiata, ope relationis (σ) suppeditat

$$\mathfrak{J}_{r+2}=(-1)^r\sum_{k=0}^{k=r+1}(-1)^k\cdot\frac{\binom{r,k+1}{\mathfrak{c}}+k\binom{r,k}{\mathfrak{c}}\varepsilon^{-(k+1)ai}}{v^{k+1}},$$

unde, comparatione coëfficientium facta, hæc prodit ipsius $\overset{(r,k)}{\mathfrak{c}}$ definitio

$$\overset{(r+1,k+1)}{\mathfrak{c}}=(k+1)\overset{(r,k+1)}{\mathfrak{c}}+k\overset{(r,k)}{\mathfrak{c}},$$

ex qua, cum formula (ε) comparata, colligitur utique

$$\overset{(r,k)}{\mathfrak{c}}=\overset{(r,k)}{\mathfrak{a}}.$$

Possumus igitur, restituto $y+\varepsilon^{-ai}$ pro v, ita $\mathfrak{J}_r$ exhibere, ut sit

$$\mathfrak{J}_r=(-1)^{r-1}\sum_{k=1}^{k=r+1}(-1)^{k-1}\cdot\frac{\overset{(r,k)}{\mathfrak{a}}}{(1+y\varepsilon^{ai})^k},$$

unde etiam, mutato ipsius a signo, prodit

$$'\mathfrak{J}_r=(-1)^{r-1}\sum_{k=1}^{k=r+1}(-1)^{k-1}\cdot\frac{\overset{(r,k)}{\mathfrak{a}}}{(1+y\varepsilon^{-ai})^k}.$$

Hinc vero, respectu ad formulas (ϱ) habito, iisdemque fere ac suprà factis reductionibus, has generales ipsorum M_r et N_r expressiones habebimus

$$\left.\begin{aligned}M_r&=(-1)^{r-1}\sum_{k=1}^{k=r+1}(-1)^{k-1}\cdot\frac{\overset{(r,k)}{\mathfrak{a}}\operatorname{Cos}k\left(\operatorname{Arc\,Tang}\frac{y\operatorname{Sin}a}{1+y\operatorname{Cos}a}\right)}{(1+2y\operatorname{Cos}a+y^2)^{\frac{k}{2}}}\\ N_r&=(-1)^{r}\sum_{k=1}^{k=r+1}(-1)^{k-1}\cdot\frac{\overset{(r,k)}{\mathfrak{a}}\operatorname{Sin}k\left(\operatorname{Arc\,Tang}\frac{y\operatorname{Sin}a}{1+y\operatorname{Cos}a}\right)}{(1+2y\operatorname{Cos}a+y^2)^{\frac{k}{2}}}\end{aligned}\right\}\ldots(v)$$

quæ, cum formulis (ι) comparatæ, relationes (π) demonstrandas utique suppeditant.

Ponamus jam $y=1$, in qua suppositione M_r et N_r in $'M_r(a)$ et $'N_r(a)$ abeant; tunc e formulis (υ) obtinebimus

$$\left.\begin{aligned}'M_r(a)&=(-1)^{r-1}\sum_{k=1}^{k=r+1}(-1)^{k-1}\cdot\frac{\overset{(r,k)}{\mathfrak{a}}\operatorname{Cos}\frac{1}{2}ka}{2^k(\operatorname{Cos}\frac{1}{2}a)^k}\\ 'N_r(a)&=(-1)^{r}\sum_{k=1}^{k=r+1}(-1)^{k-1}\cdot\frac{\overset{(r,k)}{\mathfrak{a}}\operatorname{Sin}\frac{1}{2}ka}{2^k(\operatorname{Cos}\frac{1}{2}a)^k}\end{aligned}\right\}\ldots(\varphi)$$

unde denique fit, e ratione formulæ (ν),

$$'M_{2n}(a)=0,\qquad 'N_{2n+1}(a)=0.\ldots(\chi)$$

§. 15. His præparatis, ad formulam (39) revertamur, ex qua, respectu a differentiata, adhibitis prima et secunda formularum (π), habebimus

$$\int_0^\infty\frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\cdot\frac{u^{2n}\,du}{x^2+u^2}=\frac{(-1)^n}{x}\int_0^1 y^{x-1}N_{2n}\,dy,$$

$$\int_0^\infty\frac{e^{au}+e^{-au}}{e^{\pi u}-e^{-\pi u}}\cdot\frac{u^{2n+1}\,du}{x^2+u^2}=\frac{(-1)^n}{x}\int_0^1 y^{x-1}M_{2n+1}\,dy.$$

Hinc vero, successiva per partes facta integratione, cum sit

$$'N_0(a)=\tfrac{1}{2}\operatorname{Tang}\tfrac{1}{2}a,\qquad 'M_0(a)=\tfrac{1}{2},$$

obtinemus

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} \cdot \frac{u^{2n}\,du}{x^2+u^2} =$$

$$= (-1)^n \left\{ U(a,x).x^{2n-1} - \tfrac{1}{2}\operatorname{Tang}\tfrac{1}{2}a.x^{2n-2} - {}'N_2(a).x^{2n-4} \ldots - {}'N_{2(n-1)}(a) \right\}$$

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}-e^{-\pi u}} \cdot \frac{u^{2n+1}\,du}{x^2+u^2} =$$

$$= (-1)^{n+1} \left\{ U'(a,x).x^{2n} - \tfrac{1}{2}x^{2n-1} + {}'M_1(a).x^{2n-2} + {}'M_3(a).x^{2n-4} \ldots + {}'M_{2n-1}(a) \right\}$$

positis

$$U(a,x) = \operatorname{Sin} a \int_0^1 \frac{y^x dy}{1+2y\operatorname{Cos} a + y^2},$$

$$U'(a,x) = \int_0^1 \frac{y^x (y+\operatorname{Cos} a)\,dy}{1+2y\operatorname{Cos} a + y^2},$$

inter quæ manifesto haec intercedit correlatio

$$U'(a,x) = \operatorname{Cosec} a \,.\, U(a,x+1) + \operatorname{Cotang} a\, U(a,x).$$

Jamvero supra ad formulam (39) indicavimus, $U(a,x)$ finite semper exprimi posse, et si x rationalis ponitur, quodcumque sit a, et si a in ratione commensurabili est ad π, quicumque valor ipsi x tribuatur; inde sequitur ut integralia, quæ in formulis (40) continentur, in utroque hoc casu cognita habeantur. Hæc integralia, quantum nobis cognitum est, non fuerunt antea proposita; atque revera periculum est, ut proponi potuissent, nisi beneficio relationum, quas in §. præcedente invenimus.

Si in formulis (40) $n+1$ loco n substituitur, posito

$x=0$ et $\varepsilon^{-\pi u} = y^b$, unde $u = -\frac{b}{\pi}\operatorname{Log} y$, $du = -\frac{b}{\pi}.\frac{dy}{y}$; habebimus, mutato $\frac{ba}{\pi}$ in c,

$$\left.\begin{aligned} \int_0^1 \frac{y^{c-1}-y^{-c-1}}{y^b-y^{-b}} (\operatorname{Log} y)^{2n} dy &= (-1)^n .{}'N_{2n}\left(\tfrac{c\pi}{b}\right).\left(\tfrac{\pi}{b}\right)^{2n+1} \\ \int_0^1 \frac{y^{c-1}+y^{-c-1}}{y^b-y^{-b}} (\operatorname{Log} y)^{2n+1} dy &= (-1)^n .{}'M_{2n+1}\left(\tfrac{c\pi}{b}\right).\left(\tfrac{\pi}{b}\right)^{2(n+1)} \end{aligned}\right\} \ldots (41)$$

quorum integralium valores, quamvis specie diversi, tamen cum illis revera congruunt, quos *Eulerus* [5]), aliunde procedens, iisdem adsignavit.

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§. 14. Revocemus jam priorem formulam (5), quam utrimque multiplicemus per

$$\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\,du;$$

tunc, integratione inter $u=0$ et $u=\infty$ facta, prodit

$$\int_0^\infty \frac{\psi(u)(e^{au}+e^{-au})}{e^{\pi u}+e^{-\pi u}}du = S.A_{m,n}.\varepsilon^{mx+ny} \int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}} \operatorname{Cos}(mr+ns)u.du$$

$$= S.A_{m,n}.\varepsilon^{mx+ny} \cdot \frac{\varepsilon^{\frac{1}{2}(mr+ns)}\left(1+\varepsilon^{mr+ns}\right)\operatorname{Cos}\frac{1}{2}a}{1+2\varepsilon^{mr+ns}\operatorname{Cos} a+\varepsilon^{2(mr+ns)}}.$$

---

[5]) Vide *Nov. Comm. Acad. Petropol.* Tom. XIX ann. 1774 pagg. 47 et 48.

Cum vero sit

$$\frac{(1+e^{mr+ns})\operatorname{Cos}\frac{1}{2}a}{1+2e^{mr+ns}\operatorname{Cos}a+e^{2(mr+ns)}} = \sum_{i=0}^{i=\infty}(-1)^i e^{i(mr+ns)}\operatorname{Cos}(i+\tfrac{1}{2})a,$$

Σ et S nullo modo a se invicem pendentibus, fit inde

$$\int_0^\infty \frac{\psi(u)(e^{au}+e^{-au})}{e^{\pi u}+e^{-\pi u}}du = \sum_{i=0}^{i=\infty}(-1)^i\operatorname{Cos}(i+\tfrac{1}{2})a.S.A_{m,n}.e^{m(x+\overline{i+\frac{1}{2}}.r)+n(y+\overline{i+\frac{1}{2}}.s)}$$

unde, exsistente ex formula (3)

$$S.A_{m,n}e^{m(x+\overline{i+\frac{1}{2}}.r)+n(y+\overline{i+\frac{1}{2}}.s)} = \varphi(x+\overline{i+\tfrac{1}{2}}.r,\ y+\overline{i+\tfrac{1}{2}}.s),$$

hanc generalem habebimus formulam

$$\int_0^\infty \frac{\psi(u)(e^{au}+e^{-au})}{e^{\pi u}+e^{-\pi u}}du = \sum_{i=0}^{i=\infty}(-1)^i\operatorname{Cos}(i+\tfrac{1}{2})a.\varphi(x+\overline{i+\tfrac{1}{2}}.r, y+\overline{i+\tfrac{1}{2}}.s), \quad (42)$$

ad quam eadem fere, ac supra ad formulam (6), sunt observanda.

§. 15. Jam vero ut ad speciales casus progrediamur, sit

$$\varphi(x,y) = \operatorname{Log}x,$$

unde

$$\psi(u) = \tfrac{1}{2}\operatorname{Log}(x^2+r^2u^2).$$

Posito $r=1$, quod nullo modo generalitatem minuit, obtinebimus

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\operatorname{Log}(x^2+u^2)du = 2\sum_{i=0}^{i=\infty}(-1)^i\operatorname{Cos}(i+\tfrac{1}{2})a\operatorname{Log}(x+i+\tfrac{1}{2}). \quad (43)$$

Appellemus

$$\mathfrak{L}(a,x) = \int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\operatorname{Log}(x^2+u^2)du.$$

Ex vero cognita formula

$$\frac{y\operatorname{Sin}a}{1+2y\operatorname{Cos}a+y^2} = \sum_{i=0}^{i=\infty}(-1)^{i-1}y^i\operatorname{Sin}ia$$

facili negotio hæc colligitur

$$\left(\operatorname{Sec}\tfrac{1}{2}a - \frac{2y^{x-\frac{1}{2}}(1+y)\operatorname{Cos}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\right) = 2\sum_{i=0}^{i=\infty}(-1)^i\operatorname{Cos}(i+\tfrac{1}{2})a.\left(1-y^{x+i-\frac{1}{2}}\right),$$

unde, si utrimque per $\frac{dy}{\operatorname{Log}\frac{1}{y}}$ multiplicetur, integratione inter $y=0$ et $y=1$ facta, habebimus

$$\mathfrak{L}(a,x) = \int_0^1 \left(\operatorname{Sec}\tfrac{1}{2}a - \frac{2y^{x-\frac{1}{2}}(1+y)\operatorname{Cos}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\right)\frac{dy}{\operatorname{Log}\frac{1}{y}} \quad (44)$$

Substituamus jam in formula (8) $x+\frac{1}{2}$ et $x-\frac{1}{2}$ loco x, tunc addendo simplicem hanc inter $L(a,x)$ et $\mathfrak{L}(a,x)$ correlationem invenimus

$$2\operatorname{Sin}\tfrac{1}{2}a\,\mathfrak{L}(a,x) = L(a,x+\tfrac{1}{2}) + L(a,x-\tfrac{1}{2}) \quad (45)$$

§. 16. His vero cognitis, si a in ratione qualibet commensurabili est ad π, id est $a=\frac{m\pi}{n}$, facile pro functione $\mathfrak{L}(a,x)$ invenire possumus formulas, quæ cum iis, quas supra pro $L(a,x)$ proposuimus, analogæ sunt. Etenim si ponatur in (11)

$x+\frac{1}{2}$ et $x-\frac{1}{2}$ loco x; inde addendo habebimus, ubi $m+n$ numerus impar est,

$$L(a,x+\tfrac{1}{2})+L(a,x-\tfrac{1}{2}) = 2\,\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,2n$$
$$+2\sum_{i=1}^{i=n-1}(-1)^{i-1}\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{2n}\right)}\right\}+2\sum_{i=1}^{i=n-1}(-1)^{i-1}\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$
$$=2\,\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,2n+2\sum_{i=1}^{i=n}(-1)^{i-1}\big(\mathrm{Sin}ia-\mathrm{Sin}(i-1)a\big)\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\},$$

atque, ubi $m+n$ numerus par est,

$$L(a,x+\tfrac{1}{2})+L(a,x-\tfrac{1}{2}) = 2\,\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,n$$
$$+\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}\right\}+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)}\right\}$$
$$=2\,\mathrm{Tang}\tfrac{1}{2}a\,\mathrm{Log}\,n+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\big(\mathrm{Sin}ia-\mathrm{Sin}(i-1)a\big)\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}\right\},$$

cum sit

$$\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Sin}ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)}\right\}=-\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Sin}(i-1)a\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}\right\}.$$

Hinc autem, divisione per $2\,\mathrm{Sin}\frac{1}{2}a$ facta, quia est in genere

$$\mathrm{Sin}\,ia-\mathrm{Sin}\,(i-1)\,a=2\,\mathrm{Sin}\tfrac{1}{2}a\,\mathrm{Cos}\,(i-\tfrac{1}{2})\,a,$$

atque ex formula (45)

$$\frac{L(a,x+\frac{1}{2})+L(a,x-\frac{1}{2})}{2\,\mathrm{Sin}\frac{1}{2}a}=\mathfrak{L}(a,x),$$

SPECIMEN ANALYTICUM,

THEOREMATA QUÆDAM NOVA

DE INTEGRALIBUS DEFINITIS, SUMMATIONE SERIERUM
EARUMQUE IN ALIAS SERIES TRANSFORMATIONE
EXHIBENS,

CUJUS PARTEM QUINTAM

VENIA AMPL. FAC. PHILOS. UPSAL.

P. P.

MAG. CAROLUS JOHANNES MALMSTEN
MATH. INFER. DOCENS

ET

CLAUDIUS ALBERTUS LINDHAGEN
Ostrog.

IN AUDITORIO GUSTAV. DIE XIII APR. MDCCCXLII.

H. A. M. S.

UPSALIAE,
EXCUDEBANT REGIAE ACADEMIAE TYPOGRAPHI.

colligitur denique

$$\left.\begin{aligned}\mathfrak{L}(a,x) &= \int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}} \operatorname{Log}(x^2+u^2)\,du \\ &= \operatorname{Sec}\tfrac{1}{2}a \operatorname{Log} 2n + 2\sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\} \\ &\qquad (m+n=\text{num. imp.})\end{aligned}\right\} \quad (46)$$

$$\mathfrak{L}(a,x) = \operatorname{Sec}\tfrac{1}{2}a \operatorname{Log} n + 2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}\right\}$$
$$(m+n=\text{num. par})$$

§. 17. Si in hac formula (46), posito $x = o$, faciamus

$$e^{\frac{\pi u}{n}} = y, \quad \text{unde } u = \frac{n}{\pi}\operatorname{Log} y, \quad du = \frac{n}{\pi}\cdot\frac{dy}{y};$$

habebimus utique, exsistente $a = \frac{m\pi}{n}$,

$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\left\{\operatorname{Log}\frac{n}{\pi} + \operatorname{Log}(\operatorname{Log} y)\right\}dy$$
$$= \frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n}\operatorname{Log} 2n + \frac{\pi}{n}\cdot\sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{2n}\right)}\right\}$$
$$(m+n=\text{num. imp.})$$

$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\left\{\operatorname{Log}\frac{n}{\pi} + \operatorname{Log}(\operatorname{Log} y)\right\}dy$$

$$= \frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n}\operatorname{Log} n + \frac{\pi}{n}\cdot\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(1-\frac{i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{n}\right)}\right\}.$$

$(m+n=\text{num. par})$

Hinc vero, cum sit

$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\,dy = \frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n},$$

sequitur

$$\left.\begin{aligned}&\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\operatorname{Log}(\operatorname{Log} y)\,dy\\ &= \frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n}\operatorname{Log} 2\pi + \frac{\pi}{n}\cdot\sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{2n}\right)}\right\}\end{aligned}\right\}(47)$$

$(m+n=\text{num. imp.})$

$$\left.\begin{aligned}&\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\operatorname{Log}(\operatorname{Log} y)\,dy\\ &= \frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n}\operatorname{Log}\pi + \frac{\pi}{n}\cdot\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(1-\frac{i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{n}\right)}\right\}\end{aligned}\right\}$$

$(m+n=\text{num. par})$

Posito in posteriori $m=1$, unde n numerus impar est, habebimus

$$\left.\begin{aligned}&\int_1^\infty \frac{\operatorname{Log}(\operatorname{Log} y)\,y^{n-2}\,dy}{1-y^2+y^4-y^6\ldots+y^{2(n-1)}}\\ &= \frac{\pi}{2n}\operatorname{Sec}\frac{\pi}{2n}\operatorname{Log}\pi + \frac{\pi}{n}\cdot\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{\pi}{n}\operatorname{Log}\left\{\frac{\Gamma\left(1-\frac{i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{n}\right)}\right\}\end{aligned}\right\}(48)$$

Sit ex. gr. $n=3$; mutato y^2 in y obtinemus

$$\int_1^\infty \frac{\operatorname{Log}(\operatorname{Log} y)\,dy}{1-y+y^2} = \frac{\pi}{3}\operatorname{Sec}\frac{\pi}{6}\operatorname{Log} 2\pi + \frac{2\pi}{3}\operatorname{Cos}\frac{\pi}{6}\operatorname{Log}\left\{\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})}\right\}$$

sive, cum sit

$$\Gamma(\tfrac{5}{6})\Gamma(\tfrac{1}{6}) = 2\pi,$$

$$\int_1^\infty \frac{\operatorname{Log}(\operatorname{Log} y)\,dy}{1-y+y^2} = \frac{2\pi}{\sqrt{3}}\operatorname{Log}\left(\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(\frac{1}{6})}\right).$$

§. 18. In suppositione $a = \frac{m\pi}{n}$ constat, exsistente $m+n$ numero integro impari,

$$\operatorname{Tang}\tfrac{1}{2}a = \sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Sin} ia,$$

atque, posito $i-1$ loco i

$$\operatorname{Tang}\tfrac{1}{2}a = -\sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Sin}(i-1)a,$$

unde addendo

$$2\operatorname{Tang}\tfrac{1}{2}a = \sum_{i=1}^{i=n}(-1)^{i-1}\left(\operatorname{Sin} ia - \operatorname{Sin}(i-1)a\right),$$

et, divisione per $2\operatorname{Sin}\frac{1}{2}a$ facta,

$$\operatorname{Sec}\tfrac{1}{2}a = \sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a. \quad \ldots\ldots (49)$$

Quocirca prior formularum (46) ita etiam exhiberi potest

$$\mathfrak{L}(a,x) = 2\sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a\operatorname{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{x+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\} \quad \ldots\ldots (50)$$

$(m+n=\text{num. imp.})$

Ponatur hic $x+n$ loco x, fit utique addendo

$$\mathfrak{L}(a,x+n)+\mathfrak{L}(a,x)=2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}(x+i-\tfrac{1}{2})\ldots(51)$$
$$(m+n=\text{num. imp})$$

Ceterum ex eadem formula, si $n-x$ loco x substituitur, similiter addendo prodit

$$\mathfrak{L}(a,x)+\mathfrak{L}(a,n-x)=2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{x+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$
$$+2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{n+i-x-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{n+i-x-\frac{1}{2}}{2n}\right)}\right\}$$

sive, cum sit

$$\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{n+i-x-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{n+i-x-\frac{1}{2}}{2n}\right)}\right\}$$
$$=\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{(2n)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}-\frac{x+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(1-\frac{x+i-\frac{1}{2}}{2n}\right)}\right\},$$

etiam

$$\mathfrak{L}(a,x)+\mathfrak{L}(a,n-x)=2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\mathrm{Log}\left\{\frac{2n\Gamma\left(\frac{1}{2}+\frac{x+i-\frac{1}{2}}{2n}\right)\Gamma\left(\frac{3}{2}-\frac{x+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)\Gamma\left(1-\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$

Hinc vero, e ratione cognita functionis Γ, eodem modo ac supra in §. 7 habebimus

$$\mathfrak{L}(a,x)+\mathfrak{L}(a,n-x)=2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{(i-x-\tfrac{1}{2})\mathrm{Cot}\frac{(i-x-\frac{1}{2})\pi}{2n}\right\}(52)$$
$$(m+n=\text{num. imp.})$$

Sed vero e posteriori etiam formularum (46) relationes analogas derivare possumus. Etenim, substituto ibi $x+n$ pro x, subtrahendo erit

$$\mathfrak{L}(a,x+n)-\mathfrak{L}(a,x)=2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\mathrm{Log}\left\{\frac{x+n+\frac{1}{2}-i}{x+i-\frac{1}{2}}\right\}\ldots.(53)$$
$$(m+n=\text{num. par})$$

Atque si in eadem $n-x$ loco x ponitur, fit etiam subtrahendo, calculo facto,

$$\mathfrak{L}(a,x)-\mathfrak{L}(a,n-x)=2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\mathrm{Log}\left\{\frac{x+\frac{1}{2}-i}{-x-i+\frac{1}{2}}\cdot\frac{\mathrm{Sin}\frac{(x+i-\frac{1}{2})\pi}{n}}{\mathrm{Sin}\frac{(x-i+\frac{1}{2})\pi}{n}}\right\}(54)$$
$$(m+n=\text{num. par})$$

E formulis (51) et (53), una cum (52) et (54) conjunctis, sequitur, ut, sive par sive impar sit $m+n$, functionem $\mathfrak{L}(a,x)$ pro quovis ipsius x valore cognitam habeamus, si modo per totam periodum ab $x=0$ ad $x=\frac{n}{2}$ cognita sit.

Si in (52) $x=\frac{n}{2}$ ponitur, prodit

$$\mathfrak{L}(a,\tfrac{n}{2})=\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{\left(\frac{n+1}{2}-i\right)\mathrm{Cot}\left(\frac{\pi}{4}-\frac{\overline{i-\frac{1}{2}}\cdot\pi}{2n}\right)\right\}.(55)$$
$$(m+n=\text{num. imp.})$$

unde, si simul formulam (51) consideras, colligitur, exsistente $m+n$ numero integro impari, functionem $\mathfrak{L}(a,x)$ pro $x=\frac{(2i+1)n}{2}$ per Logarithmos et functiones circulares numquam non exprimi posse. Hæc formula (55) pro $i=\frac{n+1}{2}$ expressionem $\mathrm{Log}\,0.\infty$ præsentat; sed verus ejus valor facile invenitur esse $\mathrm{Log}\left(\frac{2n}{\pi}\right)$.

§. 19. Ponatur jam successive loco x

$$x,\ x+\tfrac{2n}{r},\ x+\tfrac{4n}{r},\ x+\tfrac{6n}{r},\ \ldots\ x+\tfrac{\overline{r-1}\,.\,2n}{r}$$

in priori formularum (46), et

$$x,\ x+\tfrac{n}{r},\ x+\tfrac{2n}{r},\ x+\tfrac{3n}{r},\ \ldots x+\tfrac{\overline{r-1}\,.\,n}{r}$$

in posteriori, eodem modo, quo supra in §. 8, e cognita functionis Γ proprietate obtinebimus

$$\left.\begin{array}{l}\mathfrak{L}(a,x)+\mathfrak{L}(a,x+\frac{2n}{r})+\mathfrak{L}(a,x+\frac{4n}{r})\ldots+\mathfrak{L}(a,x+\frac{\overline{r-1}\,.\,2n}{r})\\ =r\,\mathrm{Sec}\,\frac{1}{2}a\,\mathrm{Log}\,2n+2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\frac{1}{2})a\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{r}{2}+\frac{r(x+i-\frac{1}{2})}{2n}\right)}{r^{\frac{r}{2}}\,\Gamma\left(\frac{r(x+i-\frac{1}{2})}{2n}\right)}\right\}\\ \qquad(m+n=\text{num. imp.})\\ \mathfrak{L}(a,x)+\mathfrak{L}(a,x+\frac{n}{r})+\mathfrak{L}(a,x+\frac{2n}{r})\ldots+\mathfrak{L}(a,x+\frac{\overline{r-1}\,.\,n}{r})\\ =r\,\mathrm{Sec}\,\frac{1}{2}a\,\mathrm{Log}\,n+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Cos}(i-\frac{1}{2})a\,\mathrm{Log}\left\{\frac{\Gamma\left(r+\frac{r(x-i+\frac{1}{2})}{n}\right)}{r^{r-\frac{r(2i-1)}{n}}\,\Gamma\left(\frac{r(x+i-\frac{1}{2})}{n}\right)}\right\}\\ \qquad(m+n=\text{num. par})\end{array}\right\}\quad(56)$$

unde apparet, priorem harum summarum, cum r numerus par est, finite semper per Logarithmos posse signari. Ceterum hinc sequitur, ut, exsistentibus $f'(u)$ et $f''(u)$ iisdem atque in §. 8, functionibus, fiat utique

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\,\mathrm{Log}\,f'(u)\,du =$$

$$=r\,\mathrm{Sec}\,\tfrac{1}{2}a\,\mathrm{Log}\,2n+2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{r}{2}+\frac{r(x+i-\frac{1}{2})}{2n}\right)}{r^{\frac{r}{2}}\,\Gamma\left(\frac{r(x+i-\frac{1}{2})}{2n}\right)}\right\}$$

$(m+n=\text{num. imp.})$

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\,\mathrm{Log}\,f''(u)\,du =$$

$$=r\,\mathrm{Sec}\,\tfrac{1}{2}a\,\mathrm{Log}\,n+2\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\mathrm{Cos}(i-\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{\Gamma\left(r+\frac{r(x-i+\frac{1}{2})}{n}\right)}{r^{r-\frac{r(2i-1)}{n}}\,\Gamma\left(\frac{r(x+i-\frac{1}{2})}{n}\right)}\right\}.$$

$(m+n=\text{num. par})$

§. 20. Jam vero, si permitteret hujus operis spatium, ad plures alias functionis $\varphi(x)$ formas generalem formulam (42) applicaremus. Quod cum non liceat, operæ tamen prætium fore videtur, hanc formulam cum illa (6), in §. 3 ante inventa, comparare, ut inde colligatur, qua in ratione inter se contineantur integralia

$$\int_0^\infty \frac{\psi(u)(e^{au}-e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du \quad\text{et}\quad \int_0^\infty \frac{\psi(u)(e^{au}+e^{-au})}{e^{\pi u}+e^{-\pi u}}\,du.$$

Supponamus vero simplicitatis caussa φ functionem solius x, unde igitur

$$\psi(u)=\frac{\varphi(x+ru\sqrt{-1})+\varphi(x-ru\sqrt{-1})}{2};$$

appellemus porro

$$\left.\begin{array}{l}\Pi(a,x)=\int_0^\infty \frac{\psi(u)(e^{au}-e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du\\ \Pi'(a,x)=\int_0^\infty \frac{\psi(u)(e^{au}+e^{-au})}{e^{\pi u}+e^{-\pi u}}\,du\end{array}\right\}\quad\ldots\ldots\ldots\quad(57)$$

Posito in (6) $x+\frac{r}{2}$ et deinde $x-\frac{r}{2}$ loco x, addendo erit

$$\Pi(a,x+\tfrac{r}{2})+\Pi(a,x-\tfrac{r}{2})=$$

$$=\sum_{i=0}^{i=\infty}(-1)^{i-1}\operatorname{Sin} ia\,\varphi(x+\overline{i+\tfrac{1}{2}}.r)+\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin} ia\,\varphi(x+\overline{i-\tfrac{1}{2}}.r)$$

sive, cum sit

$$\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Sin} ia\,\varphi(x+\overline{i-\tfrac{1}{2}}.r)=\sum_{i=0}^{i=\infty}(-1)^{i}\operatorname{Sin}(i+1)a\,\varphi(x+\overline{i+\tfrac{1}{2}}.r)$$

etiam

$$\Pi(a,x+\tfrac{r}{2})+\Pi(a,x-\tfrac{r}{2})=\sum_{i=0}^{i=\infty}(-1)^{i}\big(\operatorname{Sin}(i+1)a-\operatorname{Sin} ia\big)\,\varphi(x+\overline{i+\tfrac{1}{2}}.r)$$

sive denique

$$\Pi(a,x+\tfrac{r}{2})+\Pi(a,x-\tfrac{r}{2})=2\operatorname{Sin}\tfrac{1}{2}a\sum_{i=0}^{i=\infty}(-1)^{i}\operatorname{Cos}(i+\tfrac{1}{2})a\,\varphi(x+\overline{i+\tfrac{1}{2}}.r).$$

Inde vero, comparatione cum formula (42) facta, generale hoc habebimus

THEOREMA I. *Sit $\varphi(x)$ functio quædam ipsius x, et*

$$\psi(u)=\frac{\varphi(x+ru\sqrt{-1})+\varphi(x-ru\sqrt{-1})}{2},$$

designantibus $\Pi(a,x)$ et $\Pi'(a,x)$ integralia, in formula (57) proposita, hæc æque simplex ac memorabilis inter ea intercedit correlatio, ut sit

$$\Pi(a,x+\tfrac{r}{2})+\Pi(a,x-\tfrac{r}{2})=2\operatorname{Sin}\tfrac{1}{2}a\,\Pi'(a,x)\quad\ldots\ldots\quad(58)$$

Facile apparet, eandem omnino esse rationem, si φ functionem plurium variabilium supponamus. Videmus igitur relationem illam, inter $L(a,x)$ et $\mathfrak{L}(a,x)$ in formula (45) inventam, nonnisi specialem casum esse theorematis longe generalioris.

---

§. 21. His omnibus e priori formularum (5) deductis, jam ad posteriorem illarum nos convertamus. Quæ utrimque multiplicata per

$$\frac{e^{au}+e^{-au}}{e^{\pi u}-e^{-\pi u}}\,du,$$

integratione inter $u=0$ et $u=\infty$ facta, dabit

$$\int_0^\infty\frac{f(u)(e^{au}+e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du=S.A_{m,n}e^{mx+ny}\int_0^\infty\frac{e^{au}+e^{-au}}{e^{\pi u}-e^{-\pi u}}\operatorname{Sin}(mr+ns)u.\,du$$

$$=\tfrac{1}{2}S.A_{m,n}e^{mx+ny}.\frac{e^{mr+ns}-e^{-(mr+ns)}}{e^{mr+ns}+2\operatorname{Cos}a+e^{-(mr+ns)}}.$$

Cum autem sit

$$\tfrac{1}{2}.\frac{e^{mr+ns}-e^{-(mr+ns)}}{e^{mr+ns}+2\operatorname{Cos}a+e^{-(mr+ns)}}=-\tfrac{1}{2}+\sum_{i=1}^{i=\infty}(-1)^{i-1}e^{i(mr+ns)}\operatorname{Cos} ia,$$

fit etiam

$$\int_0^\infty\frac{f(u)(e^{au}+e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du=$$

$$=-\tfrac{1}{2}S.A_{m,n}e^{mx+ny}+\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Cos} ia\,S.A_{m,n}e^{m(x+ir)+n(y+is)}$$

atque denique, e ratione formulæ (3),

$$\int_0^\infty\frac{f(u)(e^{au}+e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du=-\tfrac{1}{2}\varphi(x,y)+\sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Cos} ia\,\varphi(x+ir,y+is),\quad(59)$$

ubi eadem fere, ac supra ad formulam (6), sunt observanda.

§. 22. Statuamus exempli caussa

$$\varphi(x) = \operatorname{Log} x \text{ et } r = 1;$$

tunc fit

$$f(u) = \frac{1}{2\sqrt{-1}} \operatorname{Log}\left\{\frac{x+u\sqrt{-1}}{x-u\sqrt{-1}}\right\} = \operatorname{Arc\,Tang}\frac{u}{x},$$

atque e formula (59)

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}-e^{-\pi u}} \operatorname{Arc\,Tang}\frac{u}{x}\,du = -\tfrac{1}{2}\operatorname{Log} x + \sum_{i=1}^{i=\infty}(-1)^{i-1}\operatorname{Cos} ia \operatorname{Log}(x+i). \quad (60)$$

Est vero

$$\tfrac{1}{2}\cdot\frac{1-y^2}{1+2y\operatorname{Cos} a+y^2} = \tfrac{1}{2} + \sum_{i=1}^{i=\infty}(-1)^i y^i \operatorname{Cos} ia,$$

$$\tfrac{1}{2}\cdot\frac{1-\alpha^2}{1+2\alpha\operatorname{Cos} a+\alpha^2} = \tfrac{1}{2} + \sum_{i=1}^{i=\infty}(-1)^i \alpha^i \operatorname{Cos} ia;$$

unde subtrahendo, si utrimque per $\frac{y^{x-1}dy}{\operatorname{Log}\frac{1}{y}}$ multiplicemus, integratione inter $y=0$ et $y=1$ facta, habebimus

$$\int_0^1 \left(\frac{1-\alpha^2}{1+2\alpha\operatorname{Cos} a+\alpha^2} - \frac{1-y^2}{1+2y\operatorname{Cos} a+y^2}\right)\frac{y^{x-1}dy}{\operatorname{Log}\frac{1}{y}} =$$

$$= 2\sum_{i=1}^{i=\infty}(-1)^i \operatorname{Cos} ia \int_0^1 y^{x-1}(\alpha^i - y^i)\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}}$$

et porro, si in dextero æquationis membro $y=\alpha z$ ponitur,

SPECIMEN ANALYTICUM,

THEOREMATA QUÆDAM NOVA

DE INTEGRALIBUS DEFINITIS, SUMMATIONE SERIERUM
EARUMQUE IN ALIAS SERIES TRANSFORMATIONE
EXHIBENS,

CUJUS PARTEM SEXTAM

VENIA AMPL. FAC. PHILOS. UPSAL.

P. P.

MAG. CAROLUS JOHANNES MALMSTEN
MATH. INFER. DOCENS

ET

OTTO VILHELMUS LEMKE
Ostrog.

IN AUDITORIO GUSTAV. DIE XIII APR. MDCCCXLII.

H. P. M. S.

UPSALIAE,
EXCUDEBANT REGIAE ACADEMIAE TYPOGRAPHI.

$$\int_0^1 \left(\frac{1-\alpha^2}{1+2\alpha\operatorname{Cos}a+\alpha^2} - \frac{1-y^2}{1+2y\operatorname{Cos}a+y^2}\right)\frac{y^{x-1}dy}{\operatorname{Log}\frac{1}{y}}$$

$$= 2\alpha^x \sum_{i=1}^{i=\infty}(-1)^i \alpha^i \operatorname{Cos}ia \int_0^1 \frac{z^{x-1}(1-z^i)\,dz}{\operatorname{Log}\frac{1}{z} - \operatorname{Log}\alpha}.$$

In suppositione vero α quantitatem esse, quæ unitati infinite appropinquat, sine ullo sensibili errore quantitates infinite parvas

$$\frac{1-\alpha^2}{1+2\alpha\operatorname{Cos}a+\alpha^2} \quad \text{et} \quad \operatorname{Log}\alpha$$

negligere possumus; unde fit utique

$$\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} = 2\alpha^x \sum_{i=1}^{i=\infty}(-1)^{i-1}\alpha^i \operatorname{Cos}ia \int_0^1 z^{x-1}(1-z^i)\cdot\frac{dz}{\operatorname{Log}\frac{1}{z}}$$

atque, exsistentibus

$$\int_0^1 z^{x-1}(1-z^i)\cdot\frac{dz}{\operatorname{Log}\frac{1}{z}} = \operatorname{Log}(x+i) - \operatorname{Log}x,$$

$$\sum_{i=1}^{i=\infty}(-1)^{i-1}\alpha^i \operatorname{Cos}ia = \frac{\alpha(\alpha+\operatorname{Cos}a)}{1+2\alpha\operatorname{Cos}a+\alpha^2},$$

etiam

$$\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}}$$

$$= 2\alpha^x \sum_{i=1}^{i=\infty}(-1)^{i-1}\alpha^i \operatorname{Cos}ia \operatorname{Log}(x+i) - \frac{2\alpha^{x+1}(\alpha+\operatorname{Cos}a)}{1+2\alpha\operatorname{Cos}a+\alpha^2}\operatorname{Log}x.$$

Posito jam $a = 1$, obtinebimus

$$\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\,\mathrm{Cos}a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} = -\mathrm{Log}x + 2\sum_{i=1}^{i=\infty}(-1)^{i-1}\mathrm{Cos}ia\,\mathrm{Log}(x+i), \quad (61)$$

quæ formula, cum (60) comparata, dabit

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}-e^{-\pi u}}\,\mathrm{Arc\,Tang}\frac{u}{x}\,du = \tfrac{1}{2}\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\,\mathrm{Cos}a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} \quad \ldots (62)$$

et pro $a = o$

$$\int_0^\infty \frac{2\,\mathrm{Arc\,Tang}\frac{u}{x}\,du}{e^{\pi u}-e^{-\pi u}} = \tfrac{1}{2}\int_0^1 \frac{y^{x-1}(1-y)}{1+y}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} = \tfrac{1}{2}\mathrm{Log}\frac{2}{x} + \mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+2}{2}\right)}{\Gamma\left(\frac{x+1}{2}\right)}\right\} \quad (63)$$

Hinc postremo, formula (63) a (62) et (60) subtracta, invenimus

$$\left.\begin{aligned}
&\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\,\mathrm{Arc\,Tang}\frac{u}{x}\,du = \\
&= \tfrac{1}{2}\mathrm{Log}\frac{x}{2} + \mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x+2}{2}\right)}\right\} + \tfrac{1}{2}\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\,\mathrm{Cos}a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} \\
&\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\,\mathrm{Arc\,Tang}\frac{u}{x}\,du = \\
&= -\tfrac{1}{2}\mathrm{Log}2 + \mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x+2}{2}\right)}\right\} + \sum_{i=1}^{i=\infty}(-1)^{i-1}\mathrm{Cos}\,ia\,\mathrm{Log}(x+i)
\end{aligned}\right\} \ldots (64)$$

unde consequentias satis notandas infra colligemus.

§. 22. Primum jam observare licet, posito $a = \frac{m\pi}{n}$, integrale

$$\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\,\mathrm{Cos}a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}}$$

per formulas (11) facile cognosci potest. Nam, posito $s = x+1$ et $r = x-1$, e priori concluditur

$$\mathrm{Sin}\,a\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\,\mathrm{Cos}a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} =$$

$$= \sum_{i=1}^{i=n-1}(-1)^{i-1}\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i+1}{2n}\right)}{\Gamma\left(\frac{x+i+1}{2n}\right)}\right\} - \sum_{i=1}^{i=n-1}(-1)^{i-1}\mathrm{Sin}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i-1}{2n}\right)}{\Gamma\left(\frac{x+i-1}{2n}\right)}\right\}$$

$$= \mathrm{Sin}a\,\mathrm{Log}\frac{2n}{x} + \sum_{i=1}^{i=n}(-1)^{i-1}\big(\mathrm{Sin}(i+1)a - \mathrm{Sin}(i-1)a\big)\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\},$$

unde, cum sit

$$\mathrm{Sin}\,(i+1)\,a - \mathrm{Sin}\,(i-1)\,a = 2\,\mathrm{Sin}\,a\,\mathrm{Cos}\,ia,$$

etiam

$$\left.\begin{aligned}
&\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\mathrm{Cos}a+y^2}\cdot\frac{dy}{\mathrm{Log}\frac{1}{y}} = \\
&= \mathrm{Log}\frac{2n}{x} + 2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}\,ia\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}
\end{aligned}\right\} \ldots (65)$$

$(m+n = \text{num. imp.})$

Ex vero posteriori (11), eadem substitutione facta, obtinemus

$$\operatorname{Sin} a \int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\operatorname{Cos} a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} = \sum_{i=1}^{i=\frac{n-1}{2}} (-1)^{i-1} \operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+1-i}{n}\right)\Gamma\left(\frac{x+i-1}{n}\right)}{\Gamma\left(\frac{x+n-1-i}{n}\right)\Gamma\left(\frac{x+i+1}{n}\right)}\right\},$$

unde, exsistente

$$\sum_{i=1}^{i=\frac{n-1}{2}} (-1)^{i-1} \operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+i-1}{n}\right)}{\Gamma\left(\frac{x+i+1}{n}\right)}\right\} = \sum_{i=\frac{n+1}{2}}^{i=n} (-1)^{i-1} \operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+1-i}{n}\right)}{\Gamma\left(\frac{x+n-1-i}{n}\right)}\right\},$$

fit etiam

$$\operatorname{Sin} a \int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\operatorname{Cos} a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} = \sum_{i=1}^{i=n} (-1)^{i-1} \operatorname{Sin} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+1-i}{n}\right)}{\Gamma\left(\frac{x+n-1-i}{n}\right)}\right\}$$

$$= \operatorname{Sin} a \operatorname{Log}\frac{x}{n} + \sum_{i=1}^{i=n} (-1)^{i} \left(\operatorname{Sin}(i+1)a - \operatorname{Sin}(i-1)a\right) \operatorname{Log}\Gamma\left(\frac{x+n-i}{n}\right),$$

quocirca, posito $n-i$ loco i, habebimus

$$\int_0^1 \frac{y^{x-1}(1-y^2)}{1+2y\operatorname{Cos} a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} = \operatorname{Log}\frac{x}{n} + 2\sum_{i=0}^{i=n-1} (-1)^{i} \operatorname{Cos} ia \operatorname{Log}\Gamma\left(\frac{x+i}{n}\right) \ldots (66)$$

$(m+n = \text{num. par})$

His jam præparatis, e comparatione prioris formulæ (64) cum (65) et (66) colligitur

$$\left.\begin{aligned}
&\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}} \operatorname{Arc\,Tang}\frac{u}{x}\, du = \\
&= \tfrac{1}{2}\operatorname{Log} n + \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x+2}{2}\right)}\right\} + \sum_{i=1}^{i=n} (-1)^{i-1} \operatorname{Cos} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\} \\
&\qquad (m+n = \text{num. imp.}) \\
&\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}} \operatorname{Arc\,Tang}\frac{u}{x}\, du = \\
&= \tfrac{1}{2}\operatorname{Log}\frac{n}{2} + \operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x+2}{2}\right)}\right\} + \sum_{i=1}^{i=n} (-1)^{i} \operatorname{Cos} ia \operatorname{Log}\Gamma\left(\frac{x+i}{n}\right) \\
&\qquad (m+n = \text{num. par})
\end{aligned}\right\} \ldots (67)$$

et, si quidem $x = o$ ponitur

$$\left.\begin{aligned}
&\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\, du = \frac{1}{\pi}\operatorname{Log} n\pi + \frac{2}{\pi}\cdot\sum_{i=1}^{i=n} (-1)^{i-1} \operatorname{Cos} ia \operatorname{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\} \\
&\qquad (m+n = \text{num. imp.}) \\
&\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\, du = \frac{1}{\pi}\operatorname{Log}\frac{n\pi}{2} + \frac{2}{\pi}\cdot\sum_{i=1}^{i=n-1} (-1)^{i} \operatorname{Cos} ia \operatorname{Log}\Gamma\left(\frac{i}{n}\right) \\
&\qquad (m+n = \text{num. par})
\end{aligned}\right\} \ldots (68)$$

§. 23. Ex formulis (68) jam demonstrabimus relationem memorabilem inter functiones Z' et Γ. Enimvero posito

$$e^{-\pi u} = y, \quad \text{unde } du = -\frac{1}{\pi}\cdot\frac{dy}{y},$$

fit utique

$$\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\,du = \frac{1}{\pi}\int_0^1 \frac{y^{\frac{a}{\pi}}+y^{-\frac{a}{\pi}}-2}{1-y^2}\,dy,$$

unde, cum sit in genere

$$\int_0^1 \frac{y^a-y^b}{1-y^2}\,dy = \tfrac{1}{2}Z'\left(\tfrac{b+1}{2}\right)-\tfrac{1}{2}Z'\left(\tfrac{a+1}{2}\right),$$

habebimus

$$\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\,du = \frac{1}{2\pi}\left\{2Z'\left(\tfrac{1}{2}\right)-Z'\left(\tfrac{1}{2}+\tfrac{a}{2\pi}\right)-Z'\left(\tfrac{1}{2}-\tfrac{a}{2\pi}\right)\right\}.$$

Hinc porro, exsistente

$$Z'\left(\tfrac{1}{2}-\iota\right) = Z'\left(\tfrac{1}{2}+\iota\right)-\pi\,\mathrm{Tang}\,\iota\pi,$$

et

$$Z'\left(\tfrac{1}{2}\right) = -C-2\,\mathrm{Log}\,2$$

ubi C constantem illam Euleri designat, prodit

$$\int_0^\infty \frac{e^{au}+e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\,du = \tfrac{1}{2}\mathrm{Tang}\tfrac{1}{2}a-\frac{2}{\pi}\mathrm{Log}\,2-\frac{1}{\pi}\left\{C+Z'\left(\tfrac{1}{2}+\frac{a}{2\pi}\right)\right\}, \quad (69)$$

quicumque sit valor ipsius $a < \pi$. In suppositione vero $a = \frac{m\pi}{n}$, comparatis inter se formulis (68) et (69), colligitur

$$Z'\left(\tfrac{1}{2}+\frac{m}{2n}\right) = -C-\mathrm{Log}\,4n\pi+\frac{\pi}{2}\mathrm{Tang}\frac{m\pi}{2n}-2\sum_{i=1}^{i=n}(-1)^{i-1}\mathrm{Cos}\frac{im\pi}{n}\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\}$$
$$(m+n=\text{num. imp.})$$

$$Z'\left(\tfrac{1}{2}+\frac{m}{2n}\right) = -C-\mathrm{Log}\,2n\pi+\frac{\pi}{2}\mathrm{Tang}\frac{m\pi}{2n}+2\sum_{i=1}^{i=n-1}(-1)^{i-1}\mathrm{Cos}\frac{im\pi}{n}\mathrm{Log}\,\Gamma\left(\frac{i}{n}\right)$$
$$(m+n=\text{num. par})$$

et, si quidem in priori $n-m$ et in posteriori $n-2m$ loco m ponitur,

$$Z'\left(1-\frac{m}{2n}\right) = -C-\mathrm{Log}\,4n\pi+\frac{\pi}{2}\mathrm{Cot}\frac{m\pi}{2n}+2\sum_{i=1}^{i=n}\mathrm{Cos}\frac{im\pi}{n}\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\},$$
$$(m=\text{num. imp.})$$

$$Z'\left(1-\frac{m}{n}\right) = -C-\mathrm{Log}\,2n\pi+\frac{\pi}{2}\mathrm{Cot}\frac{m\pi}{n}-2\sum_{i=1}^{i=n-1}\mathrm{Cos}\frac{2im\pi}{n}\mathrm{Log}\,\Gamma\left(\frac{i}{n}\right),$$
$$(m \text{ et } n \text{ num. quilibet})$$

unde denique, cum in genere sit

$$Z'(1-x) = Z'(x)+\pi\,\mathrm{Cotang}\,\pi x,$$

obtinebimus

$$\left.\begin{aligned} Z'\left(\frac{m}{2n}\right) &= -C-\mathrm{Log}\,4n\pi-\frac{\pi}{2}\mathrm{Cot}\frac{m\pi}{2n}+2\sum_{i=1}^{i=n}\mathrm{Cos}\frac{im\pi}{n}\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\} \\ &\qquad (m=\text{num. imp.}) \\ Z'\left(\frac{m}{n}\right) &= -C-\mathrm{Log}\,2n\pi-\frac{\pi}{2}\mathrm{Cot}\frac{m\pi}{n}-2\sum_{i=1}^{i=n-1}\mathrm{Cos}\frac{2im\pi}{n}\mathrm{Log}\,\Gamma\left(\frac{i}{n}\right). \\ &\qquad (m \text{ et } n \text{ num. quilibet}) \end{aligned}\right\} (70)$$

Harum igitur formularum beneficio functionem $Z'(x)$, pro quovis ipsius x rationali valore, per functionem Γ exprimere possumus; quæ inter has functiones correlatio haud indigna videtur attentione Mathematicorum.

Jam, si in posteriori formula (70) loco m successive ponitur $1, 2, 3, 4, \ldots\ldots n-1$, fit inde

$$\sum_{k=1}^{k=n-1}Z'\left(\frac{k}{n}\right) = -(n-1)C-(n-1)\mathrm{Log}\,2n\pi-\frac{\pi}{2}\sum_{k=1}^{k=n-1}\mathrm{Cot}\frac{k\pi}{n}-2\sum_{i=1}^{i=n-1}\mathrm{Log}\,\Gamma\left(\frac{i}{n}\right).\sum_{k=1}^{k=n-1}\mathrm{Cos}\frac{2ki\pi}{n},$$

unde, cum sit

$$\sum_{k=1}^{k=n-1} \text{Cotang}\frac{k\pi}{n} = 0, \quad \sum_{k=1}^{k=n-1} \text{Cos}\frac{2ki\pi}{n} = -1,$$

$$\sum_{i=1}^{i=n-1} \text{Log}\,\Gamma\left(\frac{i}{n}\right) = \frac{n-1}{2}\text{Log}\,2\pi - \frac{1}{2}\text{Log}\,n,$$

hanc formulam habebimus

$$Z'\left(\tfrac{1}{n}\right) + Z'\left(\tfrac{2}{n}\right) + Z'\left(\tfrac{3}{n}\right) \ldots + Z'\left(\tfrac{n-1}{n}\right) = -(n-1)C - n\,\text{Log}\,n, \ldots (71)$$

quæ ita etiam potest exhiberi

$$e^{Z'\left(\frac{1}{n}\right)} \cdot e^{Z'\left(\frac{2}{n}\right)} \cdot e^{Z'\left(\frac{3}{n}\right)} \ldots\ldots e^{Z'\left(\frac{n-1}{n}\right)} = n^{-n}\, e^{(1-n)C}.$$

Hæc formula (71), quippe cujus veritas ex hac ipsius Γ proprietate

$$\Gamma(y)\,\Gamma\left(y+\tfrac{1}{n}\right)\Gamma\left(y+\tfrac{2}{n}\right)\ldots\Gamma\left(y+\tfrac{n-1}{n}\right) = \Gamma(ny).(2\pi)^{\frac{n-1}{2}}\, n^{\frac{1}{2}-ny}$$

facile probari possit, generalem quandam præbet verificationem relationis, quæ supra inter Z' et Γ proposita est.

§. 24. Si in posteriori formularum (64) $x = o$ ponitur, comparatione cum (69) facta, sequitur

$$\sum_{i=1}^{i=\infty} (-1)^{i-1}\,\text{Cos}ia\,\text{Log}i = \frac{\pi}{4}\,\text{Tang}\,\tfrac{1}{2}a - \tfrac{1}{2}\,\text{Log}\,2\pi - \tfrac{1}{2}\left\{C + Z'\left(\tfrac{1}{2}+\frac{a}{2\pi}\right)\right\} (72)$$

atque, ubi $a = \frac{m\pi}{n}$, ope formularum (68) hanc hujus summæ expressionem habebimus

$$\left.\begin{array}{l} \sum_{i=1}^{i=\infty} (-1)^{i-1}\,\text{Cos}ia\,\text{Log}i = \frac{1}{2}\text{Log}\,2n + \sum_{i=1}^{i=n} (-1)^{i-1}\,\text{Cos}ia\,\text{Log}\left\{\dfrac{\Gamma\left(\frac{1}{2}+\frac{i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\} \\ \qquad (m+n = \text{num. imp.}) \\ \sum_{i=1}^{i=\infty} (-1)^{i-1}\,\text{Cos}ia\,\text{Log}i = \frac{1}{2}\text{Log}n + \sum_{i=1}^{i=n-1} (-1)^{i}\,\text{Cos}ia\,\text{Log}\,\Gamma\left(\frac{i}{n}\right) \\ \qquad (m+n = \text{num. par}) \end{array}\right\} (73)$$

unde redundant multæ formulæ, quas *Eulerus* in sua *Introductione in Analys. Infinit.* exhibuit. Attamen, ubi has formulas (73) ad summationem serierum applicamus et ob hanc caussam pro functione ipsa generatrice

$$\sum_{i=1}^{i=\infty} (-1)^{i-1}\,\text{Cos}ia\,\text{Log}i$$

series, quæ inde proveniunt, introducimus; magnopere est observandum, hasce series ita primum esse comparandas, ut convergentia gaudeant. Quod quidem numquam non effici potest [6]).

---

[6]) Hoc ut in promtu ponatur, sit brevitatis caussa

$$S = \sum_{i=1}^{i=\infty} (-1)^{i-1}\,\text{Cos}\,ia\,\text{Log}\,i,$$

atque primum ponamus $m+n$ numerum imparem; tunc colligitur

$$S = \sum_{k=o}^{k=\infty} \sum_{i=kn+1}^{i=(k+1)n} (-1)^{i-1}\,\text{Cos}\,ia\,\text{Log}\,i \ldots\ldots\ldots (a')$$

seu, $kn+i$ loco i substituto, exsistente $a = \frac{m\pi}{n}$,

$$S = \sum_{i=1}^{i=n} (-1)^{i-1}\,\text{Cos}\,ia \sum_{k=o}^{k=\infty} \left(\text{Log}\,(2kn+i) - \text{Log}\,\overline{(2k+1.n+i)}\right),$$

Ex. 1. Posito $a = \frac{\pi}{3}$, fit

$$\text{Log}\,2 - 2\,\text{Log}\,3 + \text{Log}\,4 + \text{Log}\,5 - 2\,\text{Log}\,6 + \text{etc.} = \text{Log}\left(\frac{3}{\pi}\,\text{Sin}\,\frac{\pi}{3}\right)$$

unde has duas expressiones facile habebimus

$$S = \sum_{i=1}^{i=n} (-1)^{i-1}\,\text{Cos}\,ia.\sum_{k=1}^{k=\infty}\left(\text{Log}\,(\overline{2k-2}.n+i) - \text{Log}\,(\overline{2k-1}.n+i)\right)$$

$$S = \sum_{i=1}^{i=n} (-1)^{i-1}\,\text{Cos}\,ia\left\{\text{Log}\,i + \sum_{k=1}^{k=\infty}\left(\text{Log}\,(2kn+i) - \text{Log}\,(\overline{2k-1}.n+i)\right)\right\},$$

quas addendo fit

$$S = \tfrac{1}{2}\sum_{i=1}^{i=n} (-1)^{i-1}\,\text{Cos}\,ia\left\{\text{Log}\,i + \sum_{k=1}^{k=\infty}\text{Log}\left(\frac{(\overline{2k-1}.n+i)^2 - n^2}{(\overline{2k-1}.n+i)^2}\right)\right\} \quad . \quad (b')$$

quæ series est convergens.

In suppositione vero, $m+n$ numerum parem esse, e formula (a') prodit, posito $kn+i$ loco i,

$$S = \sum_{i=1}^{i=n-1} (-1)^{i-1}\,\text{Cos}\,ia\sum_{k=1}^{k=\infty}\left(\text{Log}\,(\overline{k-1}.n+i) - \text{Log}\,kn\right),$$

cum sit in hoc casu

$$\text{Log}\,kn = \sum_{i=1}^{i=n-1} (-1)^{i-1}\,\text{Cos}\,ia\,\text{Log}\,kn.$$

Hinc facile habebimus

$$S = \sum_{i=1}^{i=n-1} (-1)^{i-1}\,\text{Cos}\,ia\left\{\text{Log}\,i + \sum_{k=1}^{k=\infty}\left(\text{Log}\,(kn+i) - \text{Log}\,kn\right)\right\},$$

$$S = \sum_{i=1}^{i=n-1} (-1)^{i-1}\,\text{Cos}\,ia\sum_{k=1}^{k=\infty}\left(\text{Log}\,(kn-i) - \text{Log}\,kn\right),$$

quas formulas addendo obtinemus

$$S = \tfrac{1}{2}\sum_{i=1}^{i=n-1} (-1)^{i-1}\,\text{Cos}\,ia\left\{\text{Log}\,i + \sum_{k=1}^{k=\infty}\text{Log}\left(\frac{k^2n^2 - i^2}{k^2n^2}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c')$$

et

$$\frac{2.4.5.7.8.10\ldots}{3.3.6.6.9.9\ldots} = \frac{3}{\pi}\,\text{Sin}\,\frac{\pi}{3}.$$

Ex. 2. Posito $a = \frac{2\pi}{3}$, cum sit

$$\tfrac{1}{2}\,\text{Log}\,3 + \tfrac{1}{2}\,\text{Log}\left\{\frac{3.9.9.15\ldots}{6.6.12.12\ldots}\right\} = \tfrac{1}{2}\,\text{Log}\left(\frac{6}{\pi}\right),$$

habemus

$$\text{Log}\,1 - \text{Log}\,2 - \text{Log}\,4 + \text{Log}\,5 + \text{Log}\,7 - \text{etc.} = \text{Log}\,\text{Tang}\,\frac{\pi}{6},$$

et

$$\frac{1.5.7.11.13.17\ldots}{2.4.8.10.14.16\ldots} = \text{Tang}\,\frac{\pi}{6}.$$

Ex. 3. Posito $a = \frac{\pi}{4}$, cum sit

$$\tfrac{1}{2}\,\text{Log}\,4 + \tfrac{1}{2}\,\text{Log}\left\{\frac{4.12.12.20\ldots}{8.8.16.16\ldots}\right\} = \tfrac{1}{2}\,\text{Log}\left(\frac{8}{\pi}\right),$$

sequitur

$$\text{Log}\,1 - \text{Log}\,3 - \text{Log}\,5 + \text{Log}\,7 + \text{Log}\,9 - \text{etc.} = \text{Log}\,\text{Tang}\,\frac{\pi}{8},$$

et

$$\frac{1.7.9.15.17.23\ldots}{3.5.11.13.19.21\ldots} = \text{Tang}\,\frac{\pi}{8}.$$

In formulis igitur (b') et (c') convergentes illas series invenimus, quæ, exsistente $a = \frac{m\pi}{n}$, e functione generatrice

$$S = \sum_{i=1}^{i=\infty} (-1)^{i-1}\,\text{Cos}\,ia\,\text{Log}\,i$$

proficiscuntur.

§. 25. Si formula (72) respectu a integretur, cum sit omnino

$$\int \mathrm{Tang}\tfrac{1}{2}a\,da = -2\,\mathrm{Log}\,\mathrm{Cos}\tfrac{1}{2}a,\quad \int Z'\left(\tfrac{1}{2}+\tfrac{a}{2\pi}\right)da = 2\pi\,\mathrm{Log}\,\Gamma\left(\tfrac{1}{2}+\tfrac{a}{2\pi}\right),$$

determinata constante arbitraria pro $a=0$, hanc novam habebimus formulam

$$\sum_{i=1}^{i=\infty}(-1)^{i-1}\frac{\mathrm{Sin}\,ia\,\mathrm{Log}\,i}{i} = \pi\,\mathrm{Log}\left\{\frac{\pi^{\frac{1}{2}-\frac{a}{2\pi}}}{(\mathrm{Cos}\tfrac{1}{2}a)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{a}{2\pi}\right)}\right\} - \frac{\pi}{2}(C+\mathrm{Log}\,2), \quad (74)$$

unde summam hujusce seriei

$$\frac{\mathrm{Sin}\,a\,\mathrm{Log}\,1}{1} - \frac{\mathrm{Sin}\,2a\,\mathrm{Log}\,2}{2} + \frac{\mathrm{Sin}\,3a\,\mathrm{Log}\,3}{3} - \frac{\mathrm{Sin}\,4a\,\mathrm{Log}\,4}{4} + \text{etc.}$$

per Γ possumus exprimere.

Ex. 1. Posito $a=\frac{\pi}{2}$, fit

$$\frac{\mathrm{Log}\,1}{1} - \frac{\mathrm{Log}\,3}{3} + \frac{\mathrm{Log}\,5}{5} - \text{etc.} = \frac{\pi}{4}(\mathrm{Log}\,\pi - C) - \pi\,\mathrm{Log}\,\Gamma\left(\tfrac{3}{4}\right),$$

et

$$\frac{1.5^{\frac{1}{4}}.9^{\frac{1}{9}}.13^{\frac{1}{13}}\ldots}{3^{\frac{1}{3}}.7^{\frac{1}{7}}.11^{\frac{1}{11}}.15^{\frac{1}{15}}\ldots} = \left\{\frac{\pi e^{-C}}{\left(\Gamma\left(\frac{3}{4}\right)\right)^2}\right\}^{\frac{\pi}{4}}.$$

Ex. 2. Posito $a=\frac{\pi}{3}$, fit

$$\mathrm{Sin}\,\frac{\pi}{3}\left\{\frac{\mathrm{Log}\,1}{1} - \frac{\mathrm{Log}\,2}{2} + \frac{\mathrm{Log}\,4}{4} - \frac{\mathrm{Log}\,5}{5} + \text{etc.}\right\} = \frac{\pi}{2}\mathrm{Log}\frac{\pi}{\mathrm{Sin}\frac{\pi}{3}} - \pi\,\mathrm{Log}\,\Gamma\left(\tfrac{2}{3}\right) - \frac{\pi}{6}(C+\mathrm{Log}\,2\pi)$$

atque, reductionibus rite factis,

$$\frac{\mathrm{Log}\,1}{1} - \frac{\mathrm{Log}\,2}{2} + \frac{\mathrm{Log}\,4}{4} - \frac{\mathrm{Log}\,5}{5} + \text{etc.} = \frac{\pi}{\sqrt{3}}\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right\} - \frac{\pi}{3\sqrt{3}}(C+\mathrm{Log}\,2\pi),$$

et

$$\frac{1.4^{\frac{1}{4}}.7^{\frac{1}{7}}.10^{\frac{1}{10}}\ldots}{2^{\frac{1}{2}}.5^{\frac{1}{5}}.8^{\frac{1}{8}}.11^{\frac{1}{11}}\ldots} = \left\{\frac{\Gamma\left(\frac{1}{3}\right)e^{-\frac{1}{3}C}}{(2\pi)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right\}^{\frac{\pi}{\sqrt{3}}}.$$

Ex. 3. Posito $a=\frac{2\pi}{3}$, fit

$$\frac{\mathrm{Log}\,1}{1} + \frac{\mathrm{Log}\,2}{2} - \frac{\mathrm{Log}\,4}{4} - \frac{\mathrm{Log}\,5}{5} + \frac{\mathrm{Log}\,7}{7} + \text{etc.} = \frac{\pi}{3\sqrt{3}}\mathrm{Log}\,2\pi - \frac{2\pi}{3\sqrt{3}}C - \frac{2\pi}{\sqrt{3}}\mathrm{Log}\,\Gamma\left(\tfrac{5}{6}\right),$$

et

$$\frac{1.2^{\frac{1}{2}}.7^{\frac{1}{7}}.8^{\frac{1}{8}}\ldots}{4^{\frac{1}{4}}.5^{\frac{1}{5}}.10^{\frac{1}{10}}.11^{\frac{1}{11}}\ldots} = \left\{\frac{(2\pi)^{\frac{1}{2}}e^{-\frac{1}{3}C}}{\left(\Gamma\left(\frac{5}{6}\right)\right)^2}\right\}^{\frac{\pi}{\sqrt{3}}}.$$

Ex. 4. Posito $a=\frac{\pi}{4}$, cum sit

$$\frac{\mathrm{Log}\,2}{2} - \frac{\mathrm{Log}\,6}{6} + \frac{\mathrm{Log}\,10}{10} - \frac{\mathrm{Log}\,14}{14} + \text{etc.} =$$

$$= \frac{\mathrm{Log}\,2}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\text{etc.}\right) + \frac{1}{2}\left(\frac{\mathrm{Log}\,1}{1} - \frac{\mathrm{Log}\,3}{3} + \frac{\mathrm{Log}\,5}{5} - \text{etc.}\right)$$

$$= \frac{\pi}{8}\mathrm{Log}\,2 + \frac{\pi}{8}(\mathrm{Log}\,\pi - C) - \frac{\pi}{2}\mathrm{Log}\,\Gamma\left(\tfrac{3}{4}\right),$$

sequitur

$$\mathrm{Sin}\,\frac{\pi}{4}\left\{\frac{\mathrm{Log}\,1}{1} + \frac{\mathrm{Log}\,3}{3} - \frac{\mathrm{Log}\,5}{5} - \frac{\mathrm{Log}\,7}{7} + \frac{\mathrm{Log}\,9}{9} + \text{etc.}\right\}$$

$$= \frac{\pi}{2}\,\mathrm{Log}\left(\frac{\pi}{\mathrm{Cos}\frac{\pi}{8}}\right) - \pi\,\mathrm{Log}\,\Gamma\left(\tfrac{5}{8}\right) - \frac{\pi}{2}\mathrm{Log}\,\Gamma\left(\tfrac{3}{4}\right) - \frac{\pi}{4}C,$$

unde, exsistente

$$\frac{\pi}{\mathrm{Cos}\frac{\pi}{8}} = \Gamma\left(\tfrac{3}{8}\right)\Gamma\left(\tfrac{5}{8}\right),$$

habemus

$$\frac{\text{Log}\,1}{1}+\frac{\text{Log}\,3}{3}-\frac{\text{Log}\,5}{5}-\frac{\text{Log}\,7}{7}+\text{etc.}=\frac{\pi}{\sqrt{2}}\,\text{Log}\left\{\frac{\Gamma(\frac{3}{8})}{\Gamma(\frac{5}{8})\Gamma(\frac{3}{4})}\right\}-\frac{\pi}{2\sqrt{2}}C,$$

et

$$\frac{1\,.\,3^{\frac{1}{9}}\,.\,9^{\frac{1}{9}}\,.\,11^{\frac{1}{11}}\ldots}{5^{\frac{1}{5}}\,.\,7^{\frac{1}{7}}\,.\,13^{\frac{1}{13}}\,.\,15^{\frac{1}{15}}\ldots}=\left\{\frac{\Gamma(\frac{3}{8})\,e^{-\frac{1}{8}C}}{\Gamma(\frac{5}{8})\Gamma(\frac{3}{4})}\right\}^{\frac{\pi}{\sqrt{2}}}.$$

Comparatis exemplis 1 et 2 cum formulis (37) et (38), verificationem quandam habemus formulæ (74).

§. 26. Ponamus simplicitatis caussa φ functionem solius x, et appellemus

$$M(a,x)=\int_0^\infty \frac{f(u)(e^{au}+e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du \quad \ldots\ldots\ldots \quad (75)$$

atque ut supra

$$\left.\begin{aligned}\Pi(a,x)&=\int_0^\infty \frac{\psi(u)(e^{au}-e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du\\ \Pi'(a,x)&=\int_0^\infty \frac{\psi(u)(e^{au}+e^{-au})}{e^{\pi u}+e^{-\pi u}}\,du\end{aligned}\right\}\ldots\ldots\ldots \quad (76)$$

Multiplicando formulam (6) per $\text{Sin}\frac{a}{2}$ et formulam (59) per $\text{Cos}\frac{a}{2}$, si $x-\frac{r}{2}$ loco x ponitur, fit utique, comparatione cum (42) facta,

$$\text{Cos}\tfrac{1}{2}a\,M(a,x-\tfrac{r}{2})+\text{Sin}\tfrac{1}{2}a\,\Pi(a,x-\tfrac{r}{2})=\Pi'(a,x)-\tfrac{1}{2}\text{Cos}\tfrac{1}{2}a\,\varphi(x-\tfrac{r}{2}),$$

unde, prout Π' aut Π ope formulæ (58) eliminetur, invenimus

$$\text{Cosec}\,a\,\Pi(a,x+r)+\text{Cotang}\,a\,\Pi(a,x)=\tfrac{1}{2}\varphi(x)+M(a,x),$$

$$\text{Sec}\tfrac{1}{2}a\,\Pi'(a,x+r)+\text{Sec}\tfrac{1}{2}a\,\text{Cos}\,a\,\Pi'(a,x)=$$
$$=\tfrac{1}{2}\varphi(x+\tfrac{r}{2})+\tfrac{1}{2}\varphi(x-\tfrac{r}{2})+M(a,x+\tfrac{r}{2})+M(a,x-\tfrac{r}{2}).$$

Habemus igitur hoc novum generale

THEOREMA II. *Sit φ functio quædam ipsius x, et*

$$\psi(u)=\frac{\varphi(x+ru\sqrt{-1})+\varphi(x-ru\sqrt{-1})}{2},$$
$$f(u)=\frac{\varphi(x+ru\sqrt{-1})-\varphi(x-ru\sqrt{-1})}{2\sqrt{-1}};$$

designantibus $\Pi(a,x)$, $\Pi'(a,x)$ et $M(a,x)$ integralia, in formulis (75) et (76) proposita, inter ea hæc intercedit correlatio

$$\left.\begin{aligned}\text{Cosec}\,a\,\Pi(a,x+r)+\text{Cotang}\,a\,\Pi(a,x)&=\tfrac{1}{2}\varphi(x)+M(a,x)\\ \text{Sec}\tfrac{1}{2}a\,\Pi'(a,x+r)+\text{Sec}\tfrac{1}{2}a\,\text{Cos}\,a\,\Pi'(a,x)&=\\ =\tfrac{1}{2}\varphi(x+\tfrac{r}{2})+\tfrac{1}{2}\varphi(x-\tfrac{r}{2})+M(a,x+\tfrac{r}{2})+M(a,x-\tfrac{r}{2}).&\end{aligned}\right\} \quad (77)$$

---

§. 27. Revocata jam sit posterior formula (5), quam per

$$\frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,du$$

multiplicemus. Integratione inter $u=o$ et $u=\infty$ facta, prodit

$$\int_0^\infty \frac{f(u)(e^{au}-e^{-au})}{e^{\pi u}+e^{-\pi u}}\,du=S.A_{m,n}e^{mx+ny}\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,\text{Sin}\,(mr+ns)u\,.\,du$$
$$=S.A_{m,n}e^{mx+ny}\,\frac{(e^{\frac{1}{2}(mr+ns)}-e^{-\frac{1}{2}(mr+ns)})\text{Sin}\,\frac{1}{2}a}{e^{mr+ns}+2\,\text{Cos}\,a+e^{-(mr+ns)}},$$

unde, cum sit

$$\frac{(1-e^{mr+ns})\operatorname{Sin}\frac{1}{2}a}{1+2e^{mr+ns}\operatorname{Cos}a+e^{2(mr+ns)}}=\sum_{i=o}^{i=\infty}(-1)^i e^{i(mr+ns)}\operatorname{Sin}(i+\tfrac{1}{2})a,$$

habebimus

$$\int_0^\infty \frac{f(u)(e^{au}-e^{-au})}{e^{\pi u}+e^{-\pi u}}du=-\sum_{i=o}^{i=\infty}(-1)^i\operatorname{Sin}(i+\tfrac{1}{2})a\,S.\Delta_{m,n}{}^{m(x+\overline{i+\frac{1}{2}},r)+n(y+\overline{i+\frac{1}{2}},s)}$$

sive, e ratione formulæ (3),

$$\int_0^\infty \frac{f(u)(e^{au}-e^{-au})}{e^{\pi u}+e^{-\pi u}}du=-\sum_{i=o}^{i=\infty}(-1)^i\operatorname{Sin}(i+\tfrac{1}{2})a\,\varphi(x+\overline{i+\tfrac{1}{2}},r,y+\overline{i+\tfrac{1}{2}},s)\ldots(78)$$

§. 28. Sit exempli caussa

$$\varphi(x)=\operatorname{Log}x,\text{ unde }f(u)=\operatorname{Arc\,Tang}\frac{u}{x};$$

posito $r=1$, obtinebimus

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\operatorname{Arc\,Tang}\frac{u}{x}\,du=-\sum_{i=o}^{i=\infty}(-1)^i\operatorname{Sin}(i+\tfrac{1}{2})a\operatorname{Log}(x+i+\tfrac{1}{2}).(79)$$

Constat autem

$$\frac{(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}=\sum_{i=o}^{i=\infty}(-1)^i y^i\operatorname{Sin}(i+\tfrac{1}{2})a,$$

$$\frac{(1-\alpha)\operatorname{Sin}\frac{1}{2}a}{1+2\alpha\operatorname{Cos}a+\alpha^2}=\sum_{i=o}^{i=\infty}(-1)^i\alpha^i\operatorname{Sin}(i+\tfrac{1}{2})a,$$

unde subtrahendo, si, multiplicatione per $\dfrac{y^{x-\frac{1}{2}}dy}{\operatorname{Log}\frac{1}{y}}$ facta,

inter $y=o$ et $y=1$ integremus, habebimus

$$\int_0^1\left(\frac{(1-\alpha)\operatorname{Sin}\frac{1}{2}a}{1+2\alpha\operatorname{Cos}a+\alpha^2}-\frac{(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\right)\frac{y^{x-\frac{1}{2}}dy}{\operatorname{Log}\frac{1}{y}}=$$

$$=\sum_{i=o}^{i=\infty}(-1)^i\operatorname{Sin}(i+\tfrac{1}{2})a\int_0^1 y^{x-\frac{1}{2}}(\alpha^i-y^i).\frac{dy}{\operatorname{Log}\frac{1}{y}}.$$

Quod si in dextero membro ponitur

$$y=\alpha z,\text{ unde }\operatorname{Log}\frac{1}{y}=\operatorname{Log}\frac{1}{z}-\operatorname{Log}\alpha,$$

fit utique

$$\int_0^1\left(\frac{(1-\alpha)\operatorname{Sin}\frac{1}{2}a}{1+2\alpha\operatorname{Cos}a+\alpha^2}-\frac{(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\right)\frac{y^{x-\frac{1}{2}}dy}{\operatorname{Log}\frac{1}{y}}$$

$$=\alpha^{x+\frac{1}{2}}\sum_{i=o}^{i=\infty}(-1)^i\alpha^i\operatorname{Sin}(i+\tfrac{1}{2})a\int_0^1 z^{x-\frac{1}{2}}\frac{(1-z^i)\,dz}{\operatorname{Log}\frac{1}{z}-\operatorname{Log}\alpha}.$$

Posito jam α quantitate, quæ unitati infinite appropinquat, in qua suppositione

$$\frac{(1-\alpha)\operatorname{Sin}\frac{1}{2}a}{1+2\alpha\operatorname{Cos}a+\alpha^2}\quad\text{et}\quad\operatorname{Log}\alpha$$

tamquam quantitates infinite parvas negligere possumus, obtinebimus

$$\int_0^1\frac{(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}.\frac{y^{x-\frac{1}{2}}dy}{\operatorname{Log}\frac{1}{y}}=-\alpha^{x+\frac{1}{2}}\sum_{i=o}^{i=\infty}(-1)^i\alpha^i\operatorname{Sin}(i+\tfrac{1}{2})a\int_0^1 z^{x-\frac{1}{2}}(1-z^i).\frac{dz}{\operatorname{Log}\frac{1}{z}}$$

unde, exsistentibus

$$\int_0^1 z^{x-\frac{1}{2}}(1-z^i).\frac{dz}{\operatorname{Log}\frac{1}{z}}=\operatorname{Log}(x+i+\tfrac{1}{2})-\operatorname{Log}(x+\tfrac{1}{2}),$$

$$\sum_{i=0}^{i=\infty}(-1)^i \alpha^i \operatorname{Sin}(i+\tfrac{1}{2})a = \frac{(1-\alpha)\operatorname{Sin}\frac{1}{2}a}{1+2\alpha\operatorname{Cos}a+\alpha^2},$$

colligitur denique, posito $\alpha = 1$,

$$\int_0^1 \frac{y^{x-\frac{1}{2}}(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} = -\sum_{i=0}^{i=\infty}(-1)^i \operatorname{Sin}(i+\tfrac{1}{2})a\ \operatorname{Log}(x+i+\tfrac{1}{2}).$$

Hinc, comparatione cum formula (79) facta, sequitur

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\operatorname{Arc\,Tang}\frac{u}{x}\,du = \int_0^1 \frac{y^{x-\frac{1}{2}}(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} \quad \ldots (80)$$

§. 29. Posito a in ratione qualibet commensurabili ad π, id est $a = \frac{m\pi}{n}$, integrale

$$\int_0^1 \frac{y^{x-\frac{1}{2}}(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}}$$

per formulas (11), in §. 4 ante inventas, facile cognosci potest. Quarum e priori, si quidem $r = x-\frac{1}{2}$ et $s = x+\frac{1}{2}$ ponitur, fit

$$\operatorname{Sin}a\int_0^1 \frac{y^{x-1}(1-y)}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} =$$

$$= \sum_{i=0}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{2n}\right)}\right\} - \sum_{i=1}^{i=n}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$

$$= -\sum_{i=0}^{i=n-1}(-1)^i\left(\operatorname{Sin}(i+1)a+\operatorname{Sin}ia\right)\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{2n}\right)}\right\}$$

igiturque, divisione per $2\operatorname{Cos}\frac{1}{2}a$ facta,

$$\int_0^1 \frac{y^{x-\frac{1}{2}}(1-y)\operatorname{Sin}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} = -\sum_{i=0}^{i=n-1}(-1)^i\operatorname{Sin}(i+\tfrac{1}{2})a\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n+i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{2n}\right)}\right\}. \quad (81)$$

$(m+n = \text{num. imp.})$

Eodem fere modo e posteriori colligitur

$$\operatorname{Sin}a\int_0^1 \frac{y^{x-\frac{1}{2}}(1-y)}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} =$$

$$= \sum_{i=0}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}\right\} + \sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)}\right\},$$

sive, cum sit

$$\sum_{i=1}^{i=\frac{n-1}{2}}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)}\right\} = \sum_{i=\frac{n+1}{2}}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}\right\},$$

etiam

$$\operatorname{Sin}a\int_0^1 \frac{y^{x-\frac{1}{2}}(1-y)}{1+2y\operatorname{Cos}a+y^2}\cdot\frac{dy}{\operatorname{Log}\frac{1}{y}} = \sum_{i=0}^{i=n-1}(-1)^{i-1}\operatorname{Sin}ia\operatorname{Log}\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}\right\}$$

$$= \sum_{i=0}^{i=n-1}(-1)^i\left(\operatorname{Sin}(i+1)a+\operatorname{Sin}ia\right)\operatorname{Log}\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right),$$

unde denique, divisione per $2\,\mathrm{Cos}\,\frac{1}{2}a$ facta, posito $n-i-1$ loco i, habebimus

$$\int_0^1 \frac{y^{x-\frac{1}{2}}(1-y)\,\mathrm{Sin}\,\frac{1}{2}a}{1+2y\,\mathrm{Cos}\,a+y^2}\cdot\frac{dy}{\mathrm{Log}\,\frac{1}{y}} = \sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}\,\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right) \quad ..(82)$$

$(m+n=\text{num. par})$

His jam inventis, formula (80) cum (81) et (82) comparata, dabit

$$\left.\begin{aligned}\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,\mathrm{Arc\,Tang}\,\frac{u}{x}\,du &= -\sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{x+n+i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{2n}\right)}\right\}\\ (m+n=\text{num. imp.})\\ \int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,\mathrm{Arc\,Tang}\,\frac{u}{x}\,du &= \sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}\,\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)\\ (m+n=\text{num. par})\end{aligned}\right\} ..(83)$$

et pro $x=o$

$$\left.\begin{aligned}\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,du &= -\frac{2}{\pi}\sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i+\frac{1}{2}}{2n}\right)}\right\}\\ (m+n=\text{num. imp.})\\ \int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,du &= \frac{2}{\pi}\sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}\,\Gamma\left(\frac{i+\frac{1}{2}}{n}\right).\\ (m+n=\text{num. par})\end{aligned}\right\} \ldots(84)$$

§. 30. Quicumque vero sit valor ipsius $a<\pi$, integrale

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,du$$

semper per functionem Z' assignari potest. Nam, posito

$$e^{-2\pi u}=y, \quad \text{unde } du=-\frac{1}{2\pi}\cdot\frac{dy}{y},$$

habebimus

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,du = \frac{1}{2\pi}\int_0^1 \frac{y^{-\frac{a}{2\pi}-\frac{1}{2}}-y^{\frac{a}{2\pi}-\frac{1}{2}}}{1+y}\,dy.$$

Cum vero in genere sit

$$\int_0^1 \frac{y^{r-1}\,dy}{1+y} = \tfrac{1}{2}Z'\left(\frac{r+1}{2}\right)-\tfrac{1}{2}Z'\left(\frac{r}{2}\right),$$

fit igitur

$$\int_0^1 \frac{y^{-\frac{a}{2\pi}-\frac{1}{2}}-y^{\frac{a}{2\pi}-\frac{1}{2}}}{1+y}\,dy = \tfrac{1}{2}Z'(\tfrac{3}{4}-\tfrac{a}{4\pi})+\tfrac{1}{2}Z'(\tfrac{1}{4}+\tfrac{a}{4\pi})-\tfrac{1}{2}Z'(\tfrac{3}{4}+\tfrac{a}{4\pi})-\tfrac{1}{2}Z'(\tfrac{1}{4}-\tfrac{a}{4\pi}),$$

quæ formula, e cognita relatione

$$Z'(\tfrac{1}{2}+c)-Z'(\tfrac{1}{2}-c)=\pi\,\mathrm{Tang}\,c\pi, \quad \ldots\ldots (85)$$

in hanc facile abit

$$\int_0^1 \frac{y^{-\frac{a}{2\pi}-\frac{1}{2}}-y^{\frac{a}{2\pi}-\frac{1}{2}}}{1+y}\,dy = Z'(\tfrac{1}{4}+\tfrac{a}{4\pi})-Z'(\tfrac{1}{4}-\tfrac{a}{4\pi})-\pi\,\mathrm{Tang}\,\tfrac{1}{2}a,$$

unde denique colligitur

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}+e^{-\pi u}}\,du = \frac{1}{2\pi}\left\{Z'(\tfrac{1}{4}+\tfrac{a}{4\pi})-Z'(\tfrac{1}{4}-\tfrac{a}{4\pi})\right\}-\tfrac{1}{2}\,\mathrm{Tang}\,\tfrac{1}{2}a. \quad (86)$$

Hinc in suppositione $a=\frac{m\pi}{n}$, comparatione cum (84) facta,

constat

$$\left.\begin{aligned} &Z'(\tfrac{1}{4}+\tfrac{m}{4n})-Z'(\tfrac{1}{4}-\tfrac{m}{4n})= \\ &=\pi\,\mathrm{Tang}\frac{m\pi}{2n}-4\sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}\frac{(2i+1)m\pi}{2n}\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i+\frac{1}{2}}{2n}\right)}\right\} \\ &\qquad(m+n=\text{num. imp.}) \\ &Z'(\tfrac{1}{4}+\tfrac{m}{4n})-Z'(\tfrac{1}{4}-\tfrac{m}{4n})= \\ &=\pi\,\mathrm{Tang}\frac{m\pi}{2n}+4\sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}\frac{(2i+1)m\pi}{2n}\,\mathrm{Log}\,\Gamma\left(\frac{i+\frac{1}{2}}{n}\right) \\ &\qquad(m+n=\text{num. par}) \end{aligned}\right\}(87)$$

Ponatur in illa $n-m$ et in hac $n-2m$ loco m, fit inde

$$Z'(\tfrac{1}{2}-\tfrac{m}{4n})-Z'(\tfrac{m}{4n})=\pi\,\mathrm{Cotang}\frac{m\pi}{2n}-4\sum_{i=0}^{i=n-1}\mathrm{Cos}\frac{(2i+1)m\pi}{2n}\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i+\frac{1}{2}}{2n}\right)}\right\}$$

$(m=\text{num. imp.})$

$$Z'(\tfrac{1}{2}-\tfrac{m}{2n})-Z'(\tfrac{m}{2n})=\pi\,\mathrm{Cotang}\frac{m\pi}{n}+4\sum_{i=0}^{i=n-1}\mathrm{Cos}\frac{(2i+1)m\pi}{n}\,\mathrm{Log}\,\Gamma\left(\frac{i+\frac{1}{2}}{n}\right)$$

$(m \text{ et } n \text{ num. quilibet})$

atque, e ratione formulæ (85),

$$\left.\begin{aligned} &Z'(\tfrac{1}{3}+\tfrac{m}{4n})-Z'(\tfrac{m}{4n})= \\ &=\pi\,\mathrm{Cosec}\frac{m\pi}{2n}-4\sum_{i=0}^{i=n-1}\mathrm{Cos}\frac{(2i+1)m\pi}{2n}\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i+\frac{1}{2}}{2n}\right)}\right\} \\ &\qquad(m=\text{num. imp.}) \\ &Z'(\tfrac{1}{2}+\tfrac{m}{2n})-Z'(\tfrac{m}{2n})=\pi\,\mathrm{Cosec}\frac{m\pi}{n}+4\sum_{i=0}^{i=n-1}\mathrm{Cos}\frac{(2i+1)m\pi}{n}\,\mathrm{Log}\,\Gamma\left(\frac{i+\frac{1}{2}}{n}\right) \\ &\qquad(m \text{ et } n \text{ num. quilibet}) \end{aligned}\right\}(88)$$

§. 31. Si in formula (79) $x=0$ ponitur, comparatione cum (86) facta, sequitur

$$\sum_{i=0}^{i=\infty}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}(2i+1)=\frac{\pi}{4}\mathrm{Tang}\tfrac{1}{2}a-\tfrac{1}{4}\left\{Z'(\tfrac{1}{4}+\frac{a}{4\pi})-Z'(\tfrac{1}{4}-\frac{a}{4\pi})\right\}(89)$$

atque, ubi $a=\frac{m\pi}{n}$ est, ope formularum (84) hæc summa ita potest exhiberi

$$\left.\begin{aligned} &\sum_{i=0}^{i=\infty}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}(2i+1)=\sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i+\frac{1}{2}}{2n}\right)}\right\} \\ &\qquad(m+n=\text{num. imp.}) \\ &\sum_{i=0}^{i=\infty}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}(2i+1)=-\sum_{i=0}^{i=n-1}(-1)^i\,\mathrm{Sin}(i+\tfrac{1}{2})a\,\mathrm{Log}\,\Gamma\left(\frac{i+\frac{1}{2}}{n}\right) \\ &\qquad(m+n=\text{num. par}) \end{aligned}\right\}(90)$$

Quod ad formulas (73) de serierum convergentia jam observavimus, in his etiam formulis applicandis omnino valet.

Ex. 1. Posito $a=\frac{\pi}{2}$, formula redditur, quæ in §. 24 jam ante inventa est,

$$\mathrm{Log}\,1-\mathrm{Log}\,3-\mathrm{Log}\,5+\mathrm{Log}\,7+\mathrm{Log}\,9-\text{etc.}=\mathrm{Log}\,\mathrm{Tang}\frac{\pi}{8},$$

et

$$\frac{1.7.9.15.17.23\ldots}{3.5.11.13.19.21\ldots}=\mathrm{Tang}\frac{\pi}{8}.$$

Ex. 2. Posito $a=\frac{\pi}{5}$, fit

$$\mathrm{Log}\,1-2\,\mathrm{Log}\,3+\mathrm{Log}\,5+\mathrm{Log}\,7-2\,\mathrm{Log}\,9+\text{etc.}=\mathrm{Log}\tfrac{1}{2},$$

et

$$\frac{1.5.7.11.13....}{3.3.9.9.15....}=\tfrac{1}{2}.$$

Ex. 3. Posito $a=\frac{2\pi}{5}$, fit

$$\text{Log}\,1-\text{Log}\,5-\text{Log}\,7+\text{Log}\,11+\text{Log}\,13-\text{etc.}=\text{Log}\,\text{Tang}\,\tfrac{\pi}{12},$$

et

$$\frac{1.11.13.23.25\ldots}{5.7.17.19.29\ldots}=\text{Tang}\,\tfrac{\pi}{12}.$$

§. 32. Jam vero, si formulam (89) respectu a integramus, fit inde

$$\sum_{i=0}^{i=\infty}(-1)^i\frac{\text{Cos}(i+\frac{1}{2})a\,\text{Log}(2i+1)}{2i+1}=$$

$$=\tfrac{\pi}{2}\text{Log}\left\{\Gamma(\tfrac{1}{4}+\tfrac{a}{4\pi})\Gamma(\tfrac{1}{4}-\tfrac{a}{4\pi})\right\}+\tfrac{\pi}{4}\text{Log}\,\text{Cos}\,\tfrac{1}{2}a+k$$

exsistente k constante arbitraria; quæ ut determinetur, ponamus $a=0$, unde (vid. §. 25 Ex. 1), reductione facta, sequitur

$$k=-\tfrac{\pi}{4}C-\tfrac{\pi}{2}\text{Log}\,2-\tfrac{3\pi}{4}\text{Log}\,\pi,$$

ubi C constans illa Euleri est. Substituto hoc ipsius k valore, cum sit

$$\Gamma(\tfrac{1}{4}+\tfrac{a}{4\pi})\Gamma(\tfrac{1}{4}-\tfrac{a}{4\pi})=\frac{2\pi^2}{\text{Cos}\,\frac{1}{2}a\,\Gamma(\frac{3}{4}+\frac{a}{4\pi})\Gamma(\frac{3}{4}-\frac{a}{4\pi})},$$

hanc denique habebimus formulam

$$\sum_{i=0}^{i=\infty}(-1)^i\frac{\text{Cos}(i+\frac{1}{2})a\,\text{Log}(2i+1)}{2i+1}=\tfrac{\pi}{2}\text{Log}\left\{\frac{\pi^{\frac{1}{2}}e^{-\frac{1}{2}C}}{\Gamma(\frac{3}{4}+\frac{a}{4\pi})\Gamma(\frac{3}{4}-\frac{a}{4\pi})}\right\}-\tfrac{\pi}{4}\text{Log}\,\text{Cos}\,\tfrac{1}{2}a\quad(91)$$

ex qua, quodcumque sit $a<\pi$, summam hujusce seriei

$$\text{Cos}\,\tfrac{1}{2}a\,\text{Log}\,1-\tfrac{1}{3}\text{Cos}\,\tfrac{3}{2}a\,\text{Log}\,3+\tfrac{1}{5}\text{Cos}\,\tfrac{5}{2}a\,\text{Log}\,5-\text{etc.}$$

per Γ exprimere possumus.

Ex. 1. Posito $a=\frac{\pi}{2}$, fit

$$\frac{\text{Log}\,1}{1}+\frac{\text{Log}\,3}{3}-\frac{\text{Log}\,5}{5}-\frac{\text{Log}\,7}{7}+\text{etc.}=\frac{\pi}{\sqrt{2}}\text{Log}\left\{\frac{2^{\frac{1}{4}}\pi^{\frac{1}{4}}e^{-\frac{1}{4}C}}{\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})}\right\}$$

et

$$\frac{1.3^{\frac{1}{7}}.9^{\frac{1}{7}}.11^{\frac{1}{7}}\ldots}{5^{\frac{1}{7}}.7^{\frac{1}{7}}.13^{\frac{1}{7}}.15^{\frac{1}{7}}\ldots}=\left\{\frac{2^{\frac{1}{4}}\pi^{\frac{1}{4}}e^{-\frac{1}{4}C}}{\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})}\right\}^{\frac{\pi}{\sqrt{2}}}.$$

Ex. 2. Posito $a=\frac{\pi}{3}$, fit

$$\frac{\text{Log}\,1}{1}-\frac{\text{Log}\,5}{5}+\frac{\text{Log}\,7}{7}-\frac{\text{Log}\,11}{11}+\text{etc.}=\frac{\pi}{\sqrt{3}}\text{Log}\left\{\frac{(2\pi)^{\frac{1}{2}}e^{-\frac{1}{2}C}}{5^{\frac{1}{4}}\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}\right\}$$

et

$$\frac{1.7^{\frac{1}{7}}.13^{\frac{1}{7}}.19^{\frac{1}{7}}\ldots}{5^{\frac{1}{7}}.11^{\frac{1}{7}}.17^{\frac{1}{7}}.23^{\frac{1}{7}}\ldots}=\left\{\frac{(2\pi)^{\frac{1}{2}}e^{-\frac{1}{2}C}}{3^{\frac{1}{4}}\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}\right\}^{\frac{\pi}{\sqrt{3}}}.$$

Ex. 3. Posito $a=\frac{2\pi}{5}$, cum sit

$$\frac{\text{Log}\,3}{3}-\frac{\text{Log}\,9}{9}+\frac{\text{Log}\,15}{15}-\text{etc.}=\tfrac{\pi}{12}\text{Log}\,3+\tfrac{\pi}{12}(\text{Log}\,\pi-C)-\tfrac{\pi}{3}\text{Log}\,\Gamma(\tfrac{3}{4}),$$

et

$$\Gamma(\tfrac{11}{12})\Gamma(\tfrac{7}{12})=\frac{2\pi\Gamma(\frac{3}{4})}{3^{\frac{1}{4}}\Gamma(\frac{1}{4})},$$

habemus

$$\frac{\text{Log}\,1}{1}+\frac{\text{Log}\,5}{5}-\frac{\text{Log}\,7}{7}-\frac{\text{Log}\,11}{11}+\text{etc.}=\frac{\pi}{3}\,\text{Log}\left\{\frac{3^{\frac{1}{4}}e^{-C}\Gamma(\frac{1}{4})}{2^{\frac{1}{2}}(\Gamma(\frac{3}{4}))^{3}}\right\}$$

et

$$\frac{1.5^{\frac{1}{5}}.13^{\frac{1}{13}}.17^{\frac{1}{17}}\ldots}{7^{\frac{1}{7}}.11^{\frac{1}{11}}.19^{\frac{1}{19}}.23^{\frac{1}{23}}}=\left\{\frac{3^{\frac{1}{4}}e^{-C}\Gamma(\frac{1}{4})}{2^{\frac{1}{2}}(\Gamma(\frac{3}{4}))^{3}}\right\}^{\frac{\pi}{3}}.$$

§. 33. Ponamus simplicitatis caussa φ functionem solius x, et appellemus

$$M'(a,x)=\int_0^\infty \frac{f(u)(e^{au}-e^{-au})}{e^{\pi u}+e^{-\pi u}}\,du \quad \ldots\ldots\ldots \quad (92)$$

atque ut supra

$$M(a,x)=\int_0^\infty \frac{f(u)(e^{au}+e^{-au})}{e^{\pi u}-e^{-\pi u}}\,du \quad \ldots\ldots\ldots \quad (93)$$

Posito in formula (78) $x+\frac{r}{2}$ et $x-\frac{r}{2}$ loco x, addendo erit

$$M'(a,x+\tfrac{r}{2})+M'(a,x-\tfrac{r}{2})=-\text{Sin}\tfrac{1}{2}a\,\varphi(x)+2\text{Sin}\tfrac{1}{2}a\sum_{i=1}^{i=\infty}(-1)^{i-1}\text{Cos}ia\,\varphi(x+ir),$$

unde, comparatione cum (59) facta, hoc novum habebimus

Theorema III. *Sit $\varphi(x)$ functio quaedam ipsius x, et*

$$f(u)=\frac{\varphi(x+ru\sqrt{-1})-\varphi(x-ru\sqrt{-1})}{2\sqrt{-1}},$$

designantibus $M'(a,x)$ et $M(a,x)$ integralia, in (92) et (93) proposita, inter ea haec intercedit correlatio

$$M'(a,x+\tfrac{r}{2})+M'(a,x-\tfrac{r}{2})=2\text{Sin}\tfrac{1}{2}a\,M(a,x) \quad \ldots \quad (94)$$

In suppositione φ functionem esse plurium variabilium, eandem omnino esse rationem, facile apparet.

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§. 34. His formulis jam propositis, antequam finem operi nostro imponamus, alias quasdam addere juvat, quæ ad series summandas et transformandas proxime pertinent. Ponatur igitur

$$\varphi(x)=F(v+ze^{-mx});$$

tunc erit

$$\psi(u)=\sum_{i=0}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^ie^{-imx}}{\Gamma(i+1)}\,\text{Cos}\,imru,$$

$$f(u)=-\sum_{i=0}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^ie^{-imx}}{\Gamma(i+1)}\,\text{Sin}\,imru,$$

atque e formulis (6) et (59), posito $x=o$ et $e^{-mr}=y$, habebimus

$$\left.\begin{aligned}\sum_{i=0}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^i}{\Gamma(i+1)}\cdot\frac{y^i\text{Sin}a}{1+2y^i\text{Cos}a+y^{2i}}&=\sum_{i=1}^{i=\infty}(-1)^{i-1}\text{Sin}ia\,F(v+zy^i)\\ \sum_{i=0}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^i}{\Gamma(i+1)}\cdot\frac{1-y^{2i}}{1+2y^i\text{Cos}a+y^{2i}}&=F(v+z)-2\sum_{i=1}^{i=\infty}(-1)^{i-1}\text{Cos}ia\,F(v+zy^i)\end{aligned}\right\}\quad(95)$$

quæ formulæ quatenus valeant, facile perspicitur. In suppositione, F functionem rationalem non fractam esse, patet utique, cujusvis harum formularum sinistrum membrum e finito terminorum numero constare. Tunc igitur finitam semper ha-

bebimus expressionem seriei, quæ in dextero membro exhibetur.

§. 55. Ceterum ex his formulis aliæ etiam diverso modo elici possunt. Sic, si prior per $\frac{ada}{(\mathrm{Log}\frac{1}{u})^2+a^2}$ multiplicetur, integratione inter $a=o$ et $a=\infty$ facta, prodit

$$\sum_{i=o}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^i}{\Gamma(i+1)}\cdot\frac{uy^i}{1+uy^i}=\sum_{i=1}^{i=\infty}(-1)^{i-1}u^iF(v+zy^i),$$

unde, per $\frac{du}{u}$ multiplicando atque integrando, habebimus

$$\sum_{i=o}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^i}{\Gamma(i+1)}\mathrm{Log}(1+uy^i)=\sum_{i=1}^{i=\infty}(-1)^{i-1}\cdot\frac{u^iF(v+zy^i)}{i}\ldots(96)$$

Eadem formula (95), per $2da$ multiplicata, facta integratione, dabit

$$\sum_{i=o}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^i}{\Gamma(i+1)}\mathrm{Log}(1+2y^i\mathrm{Cos}a+y^{2i})=2\sum_{i=1}^{i=\infty}(-1)^{i-1}\frac{\mathrm{Cos}iaF(v+zy^i)}{i},(97)$$

exsistente e formula (96) constante arbitraria nihilo æquali. Posito ex. gr.

$$F(v)=v^s,\quad s=\text{num. integ.},\quad v=z=1,$$

cum sit

$$\sum_{i=o}^{i=\infty}\frac{d^iF}{dv^i}=\sum_{i=o}^{i=s}\frac{\Gamma(s+1)}{\Gamma(s+1-i)},$$

sequitur

$$\sum_{i=1}^{i=\infty}(-1)^{i-1}\cdot\frac{(1+y^i)^s\mathrm{Cos}ia}{i}=\frac{1}{2}\sum_{i=o}^{i=s}\frac{\Gamma(s+1)}{\Gamma(i+1)\Gamma(s+1-i)}\cdot\mathrm{Log}(1+2y^i\mathrm{Cos}a+y^{2i}).$$

Hinc pro $a=o$, $s=1$ et $y=\frac{1}{2}$ habemus

$$\mathrm{Log}\,3=\frac{3}{2}-\frac{1}{2}\cdot\frac{5}{4}+\frac{1}{3}\cdot\frac{9}{8}-\frac{1}{4}\cdot\frac{17}{16}+\frac{1}{5}\cdot\frac{33}{32}-\text{etc.}$$

§. 56. Facili negotio e formula (97) hanc etiam habebimus

$$\sum_{i=o}^{i=\infty}\frac{a^iF}{dv^i}\cdot\frac{z^i}{\Gamma(i+1)}\mathrm{Log}\left\{\frac{1+2y^i+y^{2i}}{1+2y^i\mathrm{Cos}\,a+y^{2i}}\right\}=$$

$$=4\sum_{i=1}^{i=\infty}(-1)^{i-1}\cdot\frac{\mathrm{Sin}^2\frac{1}{2}ia\,F(v+zy^i)}{i},$$

unde per $(y^c+y^{-c})\frac{dy}{y}$ utrimque multiplicando et, posito y loco y^i, inter $y=o$ et $y=1$ integrando colligitur

$$\frac{\pi}{c}\cdot\sum_{i=o}^{i=\infty}\frac{d^iF}{dv^i}\cdot\frac{z^i}{\Gamma(i+1)}\cdot\frac{\mathrm{Sin}^2\frac{ca}{2i}}{\mathrm{Sin}\frac{c\pi}{i}}=$$

$$=\sum_{i=1}^{i=\infty}(-1)^{i-1}\frac{\mathrm{Sin}^2\frac{1}{2}ia}{i^2}\int_0^1F(v+zy)(y^{\frac{c}{i}}+y^{-\frac{c}{i}})\frac{dy}{y},(98)$$

cum sit omnino [7])

$$\int_0^1(x^a+x^{-a})\frac{dx}{x}\mathrm{Log}\left\{\frac{1+2x+x^2}{1+2x\,\mathrm{Cos}\theta+x^2}\right\}=\frac{4\pi}{a}\cdot\frac{\mathrm{Sin}^2\frac{1}{2}a\theta}{\mathrm{Sin}\,a\pi}.$$

Formula (98) $c<1$ et $a<\pi$ in genere postulat.

[7]) Vide *Legendre Exerc. de Calc. Int.* Tom. II. pag. 102.

Ex. 1. Posito

$$F(v) = \frac{v}{1+v^2},\quad v = 0,\quad z = 1,$$

tunc erit

$$\frac{d^i F}{dv^i} = (-1)^i \operatorname{Cos} \tfrac{1}{2}(i+1)\pi \Gamma(i+1),$$

atque e formula (98), si quidem in priori summa $2i+1$ loco i et $2a$ loco a ponitur, exsistente

$$\int_0^1 \frac{y^{\frac{c}{i}} + y^{-\frac{c}{i}}}{1+y^2}\,dy = \frac{\pi}{2\operatorname{Cos}\frac{c\pi}{2i}},$$

sequitur

$$\sum_{i=0}^{i=\infty} (-1)^i \frac{\operatorname{Sin}^2 \frac{ca}{2i+1}}{\operatorname{Sin}\frac{c\pi}{2i+1}} = \frac{c}{2} \cdot \sum_{i=1}^{i=\infty} (-1)^{i-1} \frac{\operatorname{Sin}^2 ia}{i^2 \operatorname{Cos}\frac{c\pi}{2i}} \quad \ldots\ldots (99)$$

Pro $a = 0$, fit inde

$$\operatorname{Sec}\frac{c\pi}{2} - \operatorname{Sec}\frac{c\pi}{4} + \operatorname{Sec}\frac{c\pi}{6} - \operatorname{Sec}\frac{c\pi}{8} + \operatorname{Sec}\frac{c\pi}{10} - \text{etc.}$$
$$= 2c\,(\operatorname{Cosec} c\pi - \tfrac{1}{9}\operatorname{Cosec}\tfrac{c\pi}{3} + \tfrac{1}{25}\operatorname{Cosec}\tfrac{c\pi}{5} - \text{etc.}).$$

Si $a = \frac{\pi}{2} - \varepsilon$ ponitur, exsistente ε quantitate indefinite parva, habebimus, posito $2c$ loco c,

$$\operatorname{Tang} c\pi - \operatorname{Tang}\tfrac{c\pi}{3} + \operatorname{Tang}\tfrac{c\pi}{5} - \operatorname{Tang}\tfrac{c\pi}{7} + \text{etc.}$$
$$= 2c\,(\operatorname{Sec} c\pi + \tfrac{1}{9}\operatorname{Sec}\tfrac{c\pi}{3} + \tfrac{1}{25}\operatorname{Sec}\tfrac{c\pi}{5} + \tfrac{1}{49}\operatorname{Sec}\tfrac{c\pi}{7} + \text{etc.}).$$

Ceterum formula (99), respectu a differentiata, dabit

$$\frac{\operatorname{Sin} ca}{\operatorname{Sin} c\pi} - \frac{\operatorname{Sin}\frac{ca}{3}}{3\operatorname{Sin}\frac{c\pi}{3}} + \frac{\operatorname{Sin}\frac{ca}{5}}{5\operatorname{Sin}\frac{c\pi}{5}} - \frac{\operatorname{Sin}\frac{ca}{7}}{7\operatorname{Sin}\frac{c\pi}{7}} + \text{etc.}$$
$$= \tfrac{1}{2}\left\{\frac{\operatorname{Sin} a}{\operatorname{Cos}\frac{c\pi}{2}} - \frac{\operatorname{Sin} 2a}{2\operatorname{Cos}\frac{c\pi}{4}} + \frac{\operatorname{Sin} 3a}{3\operatorname{Cos}\frac{c\pi}{6}} - \frac{\operatorname{Sin} 4a}{4\operatorname{Cos}\frac{c\pi}{8}} + \text{etc.}\right\}.$$

Ex. 2. Posito

$$F(v) = \frac{v}{(1+v)^2},\quad v = 0,\quad z = 1,$$

tunc erit

$$\frac{d^i F}{dv^i} = (-1)^{i+1} i\,\Gamma(i+1),$$

atque e formula (98)

$$\frac{\pi}{c} \cdot \sum_{i=1}^{i=\infty} (-1)^{i-1} \cdot \frac{i \operatorname{Sin}^2 \frac{ca}{2i}}{\operatorname{Sin}\frac{c\pi}{i}} = \sum_{i=1}^{i=\infty} (-1)^{i-1} \cdot \frac{\operatorname{Sin}^2 \frac{1}{2} ia}{i^2} \int_0^1 \frac{(y^{\frac{c}{i}} + y^{-\frac{c}{i}})}{(1+y)^2}\,dy$$

sive, cum sit [8])

$$\int_0^1 \frac{(y^{\frac{c}{i}} + y^{-\frac{c}{i}})}{(1+y)^2}\,dy = \frac{c}{i} \cdot \frac{\pi}{\operatorname{Sin}\frac{c\pi}{i}},$$

etiam

$$\frac{1}{i^2} \cdot \sum_{i=1}^{i=\infty} (-1)^{i-1} \cdot \frac{i \operatorname{Sin}^2 \frac{ca}{2i}}{\operatorname{Sin}\frac{c\pi}{i}} = \sum_{i=1}^{i=\infty} (-1)^{i-1} \cdot \frac{\operatorname{Sin}^2 \frac{1}{2} ia}{i^3 \operatorname{Sin}\frac{c\pi}{i}} \quad \ldots\ldots (100)$$

[8]) *Legendre Exerc. de Calc. Int.* Tom. II. pag. 101 form. (*g*).

Hinc pro $a = \pi - \varepsilon$, exsistente ε quantitate indefinite parva, prodit

$$\operatorname{Tang}\frac{c\pi}{2} - 2\operatorname{Tang}\frac{c\pi}{4} + 3\operatorname{Tang}\frac{c\pi}{6} - 4\operatorname{Tang}\frac{c\pi}{8} + 5\operatorname{Tang}\frac{c\pi}{10} - \text{etc.}$$
$$= 2c^2\left\{\operatorname{Cosec} c\pi + \frac{1}{3^3}\operatorname{Cosec}\frac{c\pi}{3} + \frac{1}{5^3}\operatorname{Cosec}\frac{c\pi}{5} + \text{etc.}\right\}.$$

Differentiata jam formula (100) hanc suppeditat

$$\frac{\operatorname{Sin} ca}{\operatorname{Sin} c\pi} - \frac{\operatorname{Sin}\frac{ca}{2}}{\operatorname{Sin}\frac{c\pi}{2}} + \frac{\operatorname{Sin}\frac{ca}{3}}{\operatorname{Sin}\frac{c\pi}{3}} - \frac{\operatorname{Sin}\frac{ca}{4}}{\operatorname{Sin}\frac{c\pi}{4}} + \text{etc.}$$
$$= c\left\{\frac{\operatorname{Sin} a}{\operatorname{Sin} c\pi} - \frac{1}{2^2}\cdot\frac{\operatorname{Sin} 2a}{\operatorname{Sin}\frac{c\pi}{2}} + \frac{1}{3^2}\cdot\frac{\operatorname{Sin} 3a}{\operatorname{Sin}\frac{c\pi}{3}} - \frac{1}{4^2}\cdot\frac{\operatorname{Sin} 4a}{\operatorname{Sin}\frac{c\pi}{4}} + \text{etc.}\right\}$$

quæ, denuo differentiata, dabit

$$\frac{\operatorname{Cos} ca}{\operatorname{Sin} c\pi} - \frac{1}{2}\cdot\frac{\operatorname{Cos}\frac{ca}{2}}{\operatorname{Sin}\frac{c\pi}{2}} + \frac{1}{3}\cdot\frac{\operatorname{Cos}\frac{ca}{3}}{\operatorname{Sin}\frac{c\pi}{3}} - \frac{1}{4}\cdot\frac{\operatorname{Cos}\frac{ca}{4}}{\operatorname{Sin}\frac{c\pi}{4}} + \text{etc.}$$
$$= \frac{\operatorname{Cos} a}{\operatorname{Sin} c\pi} - \frac{1}{2}\cdot\frac{\operatorname{Cos} 2a}{\operatorname{Sin}\frac{c\pi}{2}} + \frac{1}{3}\cdot\frac{\operatorname{Cos} 3a}{\operatorname{Sin}\frac{c\pi}{3}} - \frac{1}{4}\cdot\frac{\operatorname{Cos} 4a}{\operatorname{Sin}\frac{c\pi}{4}} + \text{etc.}$$

§. 37. Si, posito in posteriori formularum (95) $a = o$, $v = 1$ et $z = -1$, deinde ad eam addimus

$$\sum_{i=o}^{i=\infty} \frac{d^i F}{dv^i}\cdot\frac{(-1)^i}{\Gamma(i+1)} = F(o),$$

facile obtinemus

$$\sum_{i=1}^{i=\infty} \frac{d^i F}{dv^i}\cdot\frac{(-1)^{i-1}}{\Gamma(i+1)}\cdot\frac{1}{1+y^i} = \tfrac{1}{2}F(1) - F(o) + \sum_{i=1}^{i=\infty}(-1)^{i-1}F(1-y^i).$$

Multiplicatione jam per $y^{r-1}\,dy$ et integratione inter $y = o$ et $y = 1$ utrimque facta, prodit

$$\sum_{i=1}^{i=\infty} \frac{d^i F}{dv^i}\cdot\frac{(-1)^{i-1}}{\Gamma(i+1)}\int_0^1 \frac{y^{r-1}\,dy}{1+y^i} = \frac{\frac{1}{2}F(1) - F(o)}{r} + \sum_{i=1}^{i=\infty}(-1)^{i-1}\int_0^1 F(1-y^i)y^{r-1}\,dy,$$

atque, posito y loco y^i et præterea in dextero membro $1-y$ loco y, cum sit in genere

$$\int_0^1 \frac{y^{a-1}\,dy}{1+y} = \tfrac{1}{2}Z'\left(\tfrac{1}{2}+\tfrac{a}{2}\right) - \tfrac{1}{2}Z'\left(\tfrac{a}{2}\right),$$

fit utique

$$\sum_{i=1}^{i=\infty} \frac{d^i F}{dv^i}\cdot\frac{(-1)^{i-1}}{i\Gamma(i+1)}P\left(\tfrac{r}{2i}\right) = \frac{F(1) - 2F(o)}{r} + 2\sum_{i=1}^{i=\infty}\frac{(-1)^{i-1}}{i}\int_0^1 F(y)(1-y)^{\frac{r}{i}-1}\,dy, \quad (101)$$

exsistente

$$P\left(\tfrac{r}{2i}\right) = Z'\left(\tfrac{1}{2}+\tfrac{r}{2i}\right) - Z'\left(\tfrac{r}{2i}\right). \quad \ldots\ldots \quad (102)$$

Formulæ (101) beneficio finite jam exprimere possumus summam seriei cujusdam admodum generalis. Nam ponatur

$$F(v) = v^s \text{ et } s = \text{numerus integer},$$

tunc erit

$$\sum_{i=1}^{i=\infty} \frac{d^i F}{dv^i} = \sum_{=1}^{i=s} \frac{\Gamma(s+1)}{\Gamma(s+1-i)},$$

atque, cum sit

$$\int_0^1 y^s(1-y)^{\frac{r}{i}-1}\,dy = \frac{\Gamma(s+1)\,\Gamma\left(\frac{r}{i}\right)}{\Gamma\left(s+1+\frac{r}{i}\right)} = \frac{i^{s+1}\,\Gamma(s+1)}{r(r+i)(r+2i)\ldots(r+si)},$$

e formula (101) colligitur

$$\sum_{i=1}^{i=\infty}\frac{(-1)^{i-1}i^s}{(r+i)(r+2i)\ldots(r+si)} = -\frac{(1-2^{-s})}{2\Gamma(s+1)} + \frac{r}{2}\sum_{i=1}^{i=s}\frac{(-1)^{i-1}P\left(\frac{r}{2i}\right)}{i\Gamma(i+1)\Gamma(s+1-i)}. \quad (103)$$

Ubi r quantitas rationalis est, valor ipsius $P\left(\frac{r}{2i}\right)$ per formulas (88) facile calculari potest.

Ex. 1. Posito $s=1$ et $r=1$, fit

$$1-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\text{etc.} = \frac{3}{2}-\text{Log } 2.$$

Ex. 2. Posito $s=1$ et $r=\frac{1}{2}$, fit

$$1-\frac{1}{3}+\frac{2}{5}-\frac{3}{7}+\frac{4}{9}-\frac{5}{11}+\text{etc.} = \frac{5}{4}-\frac{\pi}{8}.$$

Ex. 3. Posito $s=2$ et $r=1$, fit

$$1-\frac{1}{2.3}+\frac{2^2}{3.5}-\frac{3^2}{4.7}+\frac{4^2}{5.9}-\text{etc.} = \frac{5}{4}+\frac{\pi}{8}-\text{Log } 2.$$

Ex. 4. Posito $s=3$ et $r=1$, fit

$$1-\frac{1}{2.3.4}+\frac{2^3}{3.5.7}+\frac{3^3}{4.7.10}-\frac{4^3}{5.9.13}+\text{et} = \frac{13}{12}-\frac{5}{9}\text{Log } 2+\frac{\pi}{2}\left(\frac{1}{4}-\frac{1}{9\sqrt{3}}\right).$$