STABLE POLYNOMIALS, RATIONAL FUNCTIONS OF SEVERAL VARIABLES, AND MULTIVARIATE OPERATOR THEORY

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Overview

Generally speaking, I have broad interests in mathematical analysis, including **complex analysis (in one and in several variables); harmonic analysis; functional analysis; probability theory and stochastic processes; and dynamical systems**, and I would be happy to advise a PhD student who wishes to work in one or more of these subfields.

Here is a selection of some specific topics I have worked on in the last few years, with salient references listed.

(1) **Stable polynomials and boundary behavior of rational functions in $\mathbb{C}^d$:**

A polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ is said to be stable with respect to some open set $\Omega \subset \mathbb{C}^d$ if p has no zeros in Ω; p may well have zeros in the closure of Ω. Such polynomials arise naturally in several contexts; for instance, they are numerators of rational functions that are holomorphic in Ω. How can boundary properties (integrability, non-tangential regularity) of such q/p be related to the geometry of how the zero set of p approaches $\partial\Omega$? Does the requirement that p be non-vanishing force p to have special algebraic properties, and can this be used to further probe rational functions having p as their denominators? We have a reasonably good understanding what the answer to these questions is in the case of (i) the two-dimensional polydisk and (ii) the two-dimensional unit ball.

The natural progression would likely be to examine (a) other two-dimensional domains, and (b) higher dimensions. In particular, examining polynomials that are stable with respect to ellipsoids $|z_1|^2 + |z_2|^{2\alpha} < 1$ ($\alpha \in \mathbb{R}$) could represent a concrete project where one could imagine exploiting ideas from the ball setting.

- (with K. Bickel and J.E. Pascoe) *Singularities of rational inner functions in higher dimensions*, Amer. J. Math. 144 (2022).
- (with K. Bickel, G. Knese, and J.E. Pascoe) *Stable polynomials and admissible numerators in product domains*, Bull. London Math. Soc. 57 (2025).
- (with G. Knese and J.E. Pascoe) *Stable polynomials and bounded rational functions in the unit ball*, https://arxiv.org/abs/2602.21051, February 2026.

(2) **Clark theory in several variables**:

A bounded holomorphic function is called inner in $\Omega \subset \mathbb{C}^d$ if its boundary values (in a suitable sense) are unimodular. Given an inner function ϕ, we consider the expression

$$\frac{1-|\phi(z)|^2}{|\alpha-\phi(z)|^2}, \quad z \in \Omega,$$

and note that this is a positive pluriharmonic function. As such, it can (for reasonable Ω) be represented as the Poisson integral of a special type of positive measure μ_ϕ which is supported on part of, or all of, $\partial\Omega$. How does the structure of this measure reflect the nature of the inner function ϕ? How do operations of ϕ translate to modifications of μ_ϕ? What can we say about the structure of the space $L^2(\mu_\sigma)$? These questions have been addressed in some detail for the subclass of rational inner functions, including in a very nice recent paper by Calzi, but detailed results are lacking for more general classes of inner functions.

Clark measures are of considerable interest in classical operator theory because of the existence of a unitary operator $J\colon (\phi H^2)^\perp \to L^2(\mu_\sigma)$ that intertwines the compressed shift on the so-called model space $(\phi H^2)^\perp$ and multiplication by z on the L^2 space of the Clark measure. In a recent paper, we have begun to develop a multi-variate extensions of this theory. For certain two-variable inner functions (in particular for a wide class of rational inner functions) we are able to identify certain infinite-dimensional perturbations of the compressed shifts that produce commuting unitaries whose Taylor spectrum coincides with the support of the corresponding Clark measure. Potential next steps include further analyzing these these perturbations (eg. to develop concrete, digestible formulas), extend the class of inner functions for which our results apply, and as usual proceed to higher dimensions/d-tuples of operators.

- (with K. Bickel and J.A. Cima) *Clark measures for rational inner functions*, Michigan Math. J. 73 (2023).
- (with J.T. Anderson, L. Bergvist, K. Bickel, and J.A. Cima) *Clark measures for rational inner functions II*, Ark. Mat. 62 (2024).
- M. Calzi, *Clark measures associated with rational inner functions on bounded symmetric domains*, 153 (2025).
- (with P. Arora, K. Bickel, and C. Liaw) *Pairs of Clark unitary operators on the bidisk and their Taylor joint spectrum*, https://arxiv.org/abs/2511.05306, November 2025.

Prerequisites

I would expect a student who wishes to work with me to have a strong background in pure mathematics in general, and in complex/harmonic/functional analysis in particular. In addition, I think it is important for students to be willing to learn about other fields in mathematics, including (but not limited to) algebraic geometry, differential geometry, dynamical systems, operator theory, etc.

Projects

In my view, it is best to try to arrive at a first project that suits the strengths and interests of the particular student, as opposed to having a fixed direction in mind from the outset. The above can be seen as some indication of my recent interests and directions where I could likely propose concrete problems at a reasonable level, but I am quite open-minded regarding suggestions from the potential student.

I strongly believe that patience, persistence, and an interest in lucid exposition are crucial in order to achieve success as a mathematician, and are in many ways more important than flashy brilliance and speed.

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