Analysis meets geometry: crystallines

Spectral theory is a branch of analysis aiming to describe connections between differential equations and their spectra. In particular spectral geometry focuses on connections between geometry of a manifold and spectra of differential operators acting on them. Spectral geometry of graphs (see [2]) describes such connections for metric graphs, which can be seen as singular one-dimensional manifolds.

It appeared that spectra of Laplacians on metric graphs described via zero sets of certain trigonometric polynomials possess remarkable properties:

- the spectrum determines the total length, number of connected components and the Euler characteristic of the graph (the number of independent cycles):
- generically the spectrum contains no arithmetic sequences;
- the spectral measure - atomic measure with the support at the eigenvalues of the Laplacian,– has Fourier transform, which is an atomic measure again.

The last property is the most remarkable: such measures, called crystalline measures, were popularized by Y.Meyer [4]; several mathematicians starting from Hamburger tried to prove that they do not exist; the first explicit example of crystalline measures came as a spectral measure for a Laplacian on a metric graph [3].

A completely new area of research emerged: instead of trying to prove that such measures do not exist, mathematicians started to investigate their properties, describe important subclasses and construct new examples in several dimentions [1]. The methods used include a beautiful interplay between analysis (Fourier analysis) and geometry (algebraic geometry), supporting that interesting mathematical results are often obtained when ideas from completely different areas of mathematics come together and interact in a non-trivial way cross-fertilising each other.

Possible projects for a PhD thesis include:

- **Spectral estimates via geometry of zero sets**: use geometric properties of zero sets of multivariate polynomials to get explicit estimates on graphs eigenvalues (both existing and new) [2].
- **Spectra of fractal metric graphs**: describe spectra and study properties of the spectral measures for infinite self-similar graphs.
- **Spectral geometry of simplicial graphs**: extend ideas used for metric graphs to study spectra of singular manifolds formed by simplicial complexes.

Necessary education: mathematically, the project lies on the border between Fourier analysis, algebraic geometry, and spectral theory, therefore knowledge in some of these areas desirable.

Mentor: **Pavel Kurasov**, contact: **kurasov@math.su.se**, http://staff.math.su.se/kurasov/

References

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