Currencies and spheres

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1 Idea

At any particular time, the market values of currencies can be considered as a point on a high-dimensional sphere. Use this geometric perspective to formulate and fit probability models for future exchange rates that do not single out any particular currency as base or quote currency. The other main novelty in this approach is the focus on currencies rather than exchange rates. There are many potential use cases for such models: forecasting, portfolio optimisation, and derivatives pricing.

2 Background

Currencies such as USD, EUR, GBP and SEK have no intrinsic value: they only have value in the sense that they can be exchanged for other goods or each other. Exchange rates denoted XXXYYY (e.g. USDSEK and EURUSD) say how much a unit of currency XXX (the base currency) costs in terms of currency YYY (the quote currency).

Exchange rates satisfy the following equations for all currencies XXX, YYY, and ZZZ.

$$\mathrm{XXXYYY} = \frac{\mathrm{XXXZZZ}}{\mathrm{YYYZZZ}} = \frac{1}{\mathrm{YYYXXX}}$$

It is *as if* each exchange rate could be written as a ratio

$$\mathrm{XXXYYY} = \frac{X}{Y}$$

where X and Y are the values of the currencies XXX and YYY in some sort of absolute sense. These values are not observable, as the two vectors of "currency values" $(X, Y, \ldots, Z)$ and $(\lambda X, \lambda Y, \ldots, \lambda Z)$ produce the same exchange rates for any $\lambda \neq 0$. In fact all vectors that produce the same exchange rates form an equivalence class (a ray from the origin in Euclidean space) and the set of all such equivalent classes constitute a projective space.

One canonical way to parameterize these projective points, or rays in Euclidean space, is by the point at which the ray intersects the unit sphere. At any given point in time the currenct exchange rates therefore correspond to a point on the unit sphere. As time goes by, the exchange rates may change and to a new point on the unit sphere. This movement from one point on the sphere to another is a *rotation*.

3 Objective

Your task is to find a reasonable probability distribution for exchange rate movements interpreted as rotations of the unit sphere. You will fit the distribution to historical data or options data.

4 Example

Let's consider the currencies $\{\mathrm{XXX}, \mathrm{YYY}, \mathrm{ZZZ}\} = \{\mathrm{EUR}, \mathrm{USD}, \mathrm{SEK}\}$.

On day 0 we have the following values with SEK as the price currency.

Pair	Rate	Currency	Intersection with sphere
EURSEK	11	EUR	$11/\sqrt{11^2 + 10^2 + 1^2} = 0.7383$
USDSEK	10	USD	$10/\sqrt{11^2 + 10^2 + 1^2} = 0.6712$
SEKSEK	1	SEK	$1/\sqrt{11^2 + 10^2 + 1^2} = 0.0671$

Note that the intersection with the sphere is independent of the choice of price currency:

Pair	Rate	Currency	Intersection with sphere
EURUSD	1.1	EUR	$1.1/\sqrt{1.1^2 + 1^2 + 0.1^2} = 0.7383$
USDUSD	1	USD	$1/\sqrt{1.1^2 + 1^2 + 0.1^2} = 0.6712$
SEKUSD	0.1	SEK	$0.1/\sqrt{1.1^2 + 1^2 + 0.1^2} = 0.0671$

If the exchange rates change, the point $(\mathrm{EUR}, \mathrm{USD}, \mathrm{SEK}) = (0.7383, 0.6712, 0.0671)$ will rotate to another point on the sphere. What's a reasonable probability distribution to fit that movement?